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THE UNIVERSITY OF ALBERTA

ANALYTICITY AND ASYMPTOTICS OF JOST FUNCTIONS

by

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ABSTRACT

The Jost functions have been so called in honour of R. Jost who presented the first detailed study of their properties in his researches (1947) in the quantum theory of potential scattering. They have since been the subject of numerous investigations, many of which define the functions for unphysical as well as physical values of their arguments.

In this work we are concerned with the Jost functions defined for general complex values of their arguments. The first chapter gives a brief description of the physical motivation for the study.

The second chapter deals with the analyticity of the Jost functions . Hitherto, it has only been possible to give analytic continuations for these functions into certain regions for potentials which have specific representations (e.g. a power series) near the scattering centre and near infinity. This study presents such continuations while only assuming continuity and integrability conditions on the potential. These continuations are greatly facilitated by a lemma which allows the use of two 'comparison' equations instead of one to construct Volterra integral equations for solutions to second order linear O.D.E.'s. We also present new contour integral techniques for establishing some old results.

The third chapter reports the partial success of an attempt to describe the Jost functions when their variables are large.

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CHAPTER I

JOST FUNCTIONS IN THE QUANTUM THEORY OF POTENTIAL SCATTERING

This chapter is devoted to a brief description of the physical problem which gives rise to the Jost functions and the necessity for a study of their analyticity and asymptotics in their arguments.

§1.1 Potential Scattering

The Schrödinger equation for a beam of particles of energy E, mass m and momentum $\underset{\sim}{P} = \hbar \underset{\sim}{k}$ moving in a region free of any scattering field is

$$\nabla^2 \psi_o + \frac{2mE}{\hbar^2} \psi_o = 0 \tag{1.1}$$

or

$$(\nabla^2 + k^2)\psi_o = 0 ,$$

where $\hbar$ is Planck's constant and $k = |\underset{\sim}{k}| = \frac{\sqrt{2mE}}{\hbar}$. Solutions of this equation represent the space-dependent part of plane waves of wavelength $\frac{2\pi}{k}$. Consider the solution $\psi_o = e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}}$ for a plane wave in the direction $\underset{\sim}{n}_o = \underset{\sim}{k}/k$. This solution represents a beam of particles in the $\underset{\sim}{n}_o$ direction whose current density is defined to be

$$\underset{\sim}{j}_o = \frac{\hbar}{2mi}(\psi_o^* \nabla \psi_o - \psi_o \nabla \psi_o^*) = \frac{\hbar \underset{\sim}{k}}{m} . \tag{1.2}$$

In the case of a similar beam incident on a scattering potential $U(\underset{\sim}{r})$ the Schrödinger equation is

$$\nabla^2 \psi + \frac{2m}{\hbar^2}(E - U(\underset{\sim}{r}))\psi = 0 \tag{1.3}$$

or

$$(\nabla^2 + k^2 - V(\underset{\sim}{r}))\psi = 0$$

where $V(\underset{\sim}{r}) = \frac{2m}{\hbar^2} U(\underset{\sim}{r})$. Under the conditions that $V(\underset{\sim}{r})$ be measurable and $|V(\underset{\sim}{r})| < F(r)$, $r = |\underset{\sim}{r}|$, where

$$\int_0^\infty rF(r)dr < \infty$$

and

$$\int_0^\infty r^2F(r)dr < \infty ,$$

Hunziker* has discussed the existence and uniqueness of a bounded solution to (1.3) with the asymptotic behaviour

(1.4) $$\psi(\underset{\sim}{r}) = e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}} + e^{ikr}/r \, A(\theta,\varphi) + o(1/r) , \text{ as } r \to \infty ,$$

where r, θ, φ are spherical polar co-ordinates. If the potential is central, i.e. $V(\underset{\sim}{r}) = V(r)$, and the polar axis is taken to be $\underset{\sim}{n}_o$ then (1.4) may be written

(1.5) $$\psi(\underset{\sim}{r}) = e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}} + e^{ikr}/r \, A(\theta) + o(1/r), \text{ as } r \to \infty .$$

We identify the two terms in (1.5) to be

$$\psi_o(\underset{\sim}{r}) = e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}} ,$$

the incident plane wave, and

$$\psi_s(\underset{\sim}{r}) = e^{ikr}/r \, A(\theta),$$

a scattered spherical wave. The current density in the plane wave is given by (1.2) and in the scattered wave by

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* Hunziker, W., "Regularitatseigenschaften der Streuamplitude im Fall der Potentialstreuuing", Helv. Phys. Acta 34, p. 593 (1961).

$$\underset{\sim}{j}_s = \frac{\hbar}{2mi}(\psi_s^* \nabla \psi_s - \psi_s \nabla \psi_s^*) \tag{1.6}$$

$$= \frac{\hbar k}{m} \frac{|A(\theta)|^2}{r^3} \underset{\sim}{r} .$$

Hence the number of particles crossing an element dS of a sphere of radius r, centre $\underset{\sim}{r} = 0$, per unit time is

$$\frac{\hbar k}{m} \frac{|A(\theta)|^2}{r^2} dS ,$$

and the number of particles per unit time scattered into a solid angle element $d\Omega$ in the direction θ is

$$\frac{\hbar k}{m} |A(\theta)|^2 d\Omega .$$

From (1.2) the number of particles per unit of incident current scattered into $d\Omega$ per unit time is therefore

$$|A(\theta)|^2 d\Omega .$$

The quantity $|A(\theta)|^2$ is called the differential scattering cross-section and $A(\theta)$ is called the scattering amplitude.

§1.2 Jost Functions and the Scattering Amplitude

One approach to determining the scattering amplitude is, formally, as follows. The functions $\psi(\underset{\sim}{r}) = \psi(r, \theta)$ and $A(\theta) = A(k, \cos\theta)$ are expanded as series of Legendre polynomials

$$\psi(r, \theta) = \sum_{\ell=0}^{\infty} \frac{\varphi_\ell(k,r)}{r} P_\ell(\cos\theta) \tag{1.7}$$

$$A(k, \cos\theta) = \sum_{\ell=0}^{\infty} a_\ell(k) P_\ell(\cos\theta) . \tag{1.8}$$

The function $\varphi_\ell(k;r)$ must satisfy the differential equation

$$\frac{d^2\varphi_\ell}{dr^2} + \left(k^2 - \frac{\ell(\ell+1)}{r^2} - V(r)\right)\varphi_\ell = 0 \tag{1.9}$$

and φ_ℓ must be bounded and have a continuous bounded derivative $0 \leq r < \infty$. The equation (1.9) will be referred to as the radial Schrodinger equation. It will be proved later (Theorem 2.1) that if $\int_{R \geq 0}^{\infty} V(r)dr < \infty$ then there are two linearly independent solutions to (1.9) which behave like $e^{\pm ikr}$, as $r \to \infty$, respectively. Hence, as $r \to \infty$, any solution $\varphi_\ell(k,r)$ to (1.9) has the behaviour

$$\varphi_\ell(k;r) = B(\ell,k)e^{i(kr-\frac{1}{2}\ell\pi-\pi/2)}(1+o(1)) + C(\ell,k)e^{-i(kr-\frac{1}{2}\ell\pi-\pi/2)}(1+o(1)). \tag{1.10}$$

From the known* Legendre polynomial development of $e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}}$ we have

$$\begin{aligned} & e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}} + e^{ikr}/r\, A(k,\cos\theta) \\ & = \sum_{\ell=0}^{\infty} \left[(2\ell+1)e^{i\ell\pi/2}\left(\frac{\pi}{2kr}\right)^{\frac{1}{2}} J_{\ell+\frac{1}{2}}(kr) + a_\ell(k)e^{ikr}/r\right]P_\ell(\cos\theta) \end{aligned} \tag{1.11}$$

and, since

$$J_{\ell+\frac{1}{2}}(kr) = \left(\frac{2}{\pi kr}\right)^{\frac{1}{2}}\cos(kr - \tfrac{1}{2}\ell\pi - \pi/2)(1 + O(1/r)),$$

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* Antosiewicz, H. A., "Bessel Functions of Fractional Order", U.S. Dept. of Comm., Natl. Bureau of Standards App. Math. Ser. 55, p. 445, (1964).

as $r \to \infty$, it follows from (1.3) that

$$(1.12) \qquad r\psi(r,\theta) = \sum_{\ell=0}^{\infty} \left[\frac{2\ell+1}{2k} e^{-i(kr-\ell\pi-\pi/2)}(1+o(1)) + \left(\frac{2\ell+1}{2k} e^{-i\pi/2} + a_\ell(k)\right)e^{ikr}(1+o(1)\right] P_\ell(\cos\theta),$$

and hence

$$(1.13) \qquad \varphi_\ell(k;r) = \frac{2\ell+1}{2k} e^{-i(kr-\ell\pi-\pi/2)}(1+o(1)) + \left(\frac{2\ell+1}{2k} e^{-i\pi/2}+a_\ell(k)\right)e^{ikr}(1+o(1)), \quad \text{as } r \to \infty .$$

Comparison of (1.10) and (1.13) yields

$$(1.14) \qquad \frac{B(\ell,k)}{C(\ell,k)} = 1 + \frac{2ik}{2\ell+1} a_\ell(k)$$

i.e.
$$a_\ell(k) = \frac{2\ell+1}{2ik}\left[\frac{B(\ell,k)}{C(\ell,k)} - 1\right]$$

and from (1.8) the scattering amplitude $A(k,\cos\theta)$ is given by

$$(1.15) \qquad A(k,\cos\theta) = \sum_{\ell=0}^{\infty} \frac{2\ell+1}{2ik}\left[\frac{B(\ell,k)}{C(\ell,k)} - 1\right] P_\ell(\cos\theta) .$$

Since only the ratio $\frac{B}{A}$ is important the solution φ_ℓ to (1.9) need only be defined to within a constant multiple. The particular φ_ℓ which we will study will be given in the next chapter.

In what follows it will be convenient to take $\lambda = \ell + 1/2$ and to work in terms of this new parameter. Equation (1.9) may be written

$$(1.16) \qquad \frac{d^2\varphi}{dr^2} + \left(k^2 - \frac{\lambda^2 - 1/4}{r^2} - V(r)\right)\varphi = 0 ,$$

and the Jost functions $f^{(1)}(\lambda,k)$ and $f^{(2)}(\lambda,k)$ are then defined to be

(1.17) $$f^{(1)}(\lambda,k) = (2k\pi)^{1/2} C(\ell,k)$$

and

$$f^{(2)}(\lambda,k) = (2k\pi)^{1/2} B(\ell,k) ;$$

also $a(\lambda,k)$ is defined

(1.18) $$a(\lambda,k) = \frac{\lambda}{ik}\left[\frac{f^{(2)}(\lambda,k)}{f^{(1)}(\lambda,k)} - 1\right],$$

so that, from (1.15),

(1.19) $$A(k,z) = \sum_{\ell=0}^{\infty} a(\lambda,k)P_{\lambda-1/2}(z)), \quad \lambda = \ell + 1/2, \quad z = \cos\theta$$

§1.3 Jost Functions with Complex Arguments

Although $f^{(1,2)}(\lambda,k)$ have physical meaning only when $\lambda = \ell + 1/2$, $\ell = 0, 1, 2, \ldots$, and k is purely real or purely imaginary (so that the energy $E = \frac{\hbar^2k^2}{2m}$ is real) physicists have found it useful to extend the definition of these functions to complex values of λ and k. This is usually accomplished by taking ℓ and k complex in equation (1.9) and to define the Jost functions by the asymptotic expression (1.10) as before. The questions of the analyticity and asymptotics of the Jost functions in their arguments arise mainly because researchers have found useful expressions for physically important functions in terms of contour integrals and Poisson integrals involving the Jost functions in the complex λ and k planes. We give two examples, namely dispersion relations in the k-plane, and the Watson transform in the λ-plane.

Dispersion relations are a simple consequence of Cauchy's integral formula. If $w(k)$ is a regular analytic function in $\mathrm{Im}k \geq 0$ then

$$(1.20) \qquad w(k) = \frac{1}{2\pi i}\int_C \frac{w(k')}{k'-k}\,dk', \qquad \mathrm{Im}k > 0$$

where C is any simple closed contour about $k' = k$. In particular C may be taken to be an interval of length $2R$ of the real k'-axis and a semicircle of radius R in the upper k'-plane and enclosing $k' = k$. If $w(k') \to 0$, as $k' \to \infty$, uniformly with respect to $\arg k'$, $0 \leq \arg k' \leq \pi$ then

$$(1.21) \qquad w(k) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{w(k')}{k'-k}\,dk', \qquad \mathrm{Im}k > 0 .$$

If k is real then

$$(1.22) \qquad w(k) = \lim_{\eta \to 0+} \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{w(k')}{k'-k-i\eta}\,dk'$$

$$= \frac{1}{\pi i}\, P\int_{-\infty}^{\infty} \frac{w(k')}{k'-k}\,dk' .$$

If $w(k) = u(k) + iv(k)$ (u, v real) then it follows from (1.22) that

$$(1.23) \qquad u(k) = \frac{1}{\pi}\, P\int_{-\infty}^{\infty} \frac{v(k')dk'}{k'-k}$$

and

$$(1.24) \qquad v(k) = -\frac{1}{\pi}\, P\int_{-\infty}^{\infty} \frac{u(k')dk'}{k'-k} .$$

Expressions like (1.23) and (1.24) are called dispersion relations, and if they can be written for the Jost functions for example, they are important in that they affect physically measurable properties

of the scattering without involving details of the potential. The reader is referred to the book of Goldberger and Watson [28] for a discussion of dispersion relations.

The Watson transform is pertinent to the expression (1.19) for the scattering amplitude $A(k,z)$ which may be written

(1.25) $$A(k,z) = \frac{-1}{2i} \int_{(\frac{1}{2}+)}^{\infty} \frac{a(\lambda,k)}{\cos \pi\lambda} P_{\lambda-\frac{1}{2}}(-z) d\lambda$$

provided $f^{(1,2)}(\lambda,k)$ are analytic functions of λ for fixed k and equal to the physical Jost functions at $\lambda = \ell + \frac{1}{2}$, $\ell = 0, 1, 2 \ldots$. Here the symbol $\int_{(\frac{1}{2}+)}^{\infty}$ denotes a contour of integration which begins at ∞ and ends at $\infty e^{2i\pi}$ encircling $\lambda = \frac{1}{2}$ in the positive sense. If $a(\lambda,k) \to 0$ sufficiently rapidly as $\lambda \to \infty$ in $\mathrm{Re}\lambda \geq 0$ and is analytic there then the contour in (1.25) may be deformed to $\mathrm{Re}\lambda = 0$ so that one finds a new expression for $A(k,z)$:

(1.26) $$A(k,z) = \frac{1}{2i} \int_{-i\infty}^{i\infty} \frac{a(\lambda,k)}{\cos \pi\lambda} P_{\lambda-\frac{1}{2}}(-z) d\lambda + \sum_{\mathrm{Re}\lambda_n > 0} \frac{\pi\alpha(\lambda_n,k)}{\cos \pi\lambda_n} P_{\lambda_n-\frac{1}{2}}(-z)$$

where $a(\lambda,k)$ has poles of residue $\alpha(\lambda_n,k)$ at $\lambda = \lambda_n$. This process of deforming the contour is called the Watson transform.

The Legendre polynomial development (1.15) of $A(k,z)$ converges in an ellipse in the complex z-plane of foci ± 1 and semi-axes $\frac{1}{2}|p \pm \frac{1}{p}|$ where*

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* Whittaker, E. T., and Watson, G. N., "A Course of Modern Analysis", Camb. Univ. Press, p. 323 (1962).

(1.27) $$P = \lim_{\ell \to \infty} |a_\ell(k)|^{-1/\ell} .$$

The main purpose of the Watson transform is that it gives an analytic continuation of $A(k,z)$ out of this ellipse. For $z \to \infty$

$$P_\lambda(-z) = \frac{2^{-\lambda-1}\pi^{-\frac{1}{2}}\Gamma(-\frac{1}{2}-\lambda)}{\Gamma(-\lambda)}(-z)^{-\lambda-1}(1 + O(z^{-2}))$$
$$+ \frac{2^\lambda\Gamma(\frac{1}{2}+\lambda)}{\Gamma(1+\lambda)}(-z)^\lambda(1 + O(z^{-2}))^*$$

so that the integral in (1.26) is $O(z^{-\frac{1}{2}})$ and the main contribution to $A(k,z)$ comes from λ_N, the pole of $a(\lambda,k)$ farthest to the right in the λ-plane, if these poles are bounded by some line $\mathrm{Re}\lambda$ = constant, so that

(1.28) $$A(k,z) = \frac{\pi\alpha(\lambda_N,k)2^{\lambda_N-\frac{1}{2}}\Gamma(\lambda_N)}{\cos\pi\lambda_N\Gamma(\frac{1}{2}+\lambda_N)}(-z)^{\lambda_N-\frac{1}{2}}(1 + o(1))$$

as $z \to \infty$.

For further applications of the Watson transform the interested reader is referred to the report of N. D. Kazarinoff to the 1964 Langer Symposium on asymptotics.**

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* Stegun, I. A., "Legendre Functions", U.S. Dept. of Comm., Natl. Bureau of Standards App. Math. Ser. 55, p. 332 (1964).

** Wilcox, C. H., "Asymptotic Solutions of Differential Equations and their Applications", John Wiley and Sons, Inc., pp. 231-243 (1964).

CHAPTER II

THE ANALYTICITY OF THE JOST FUNCTIONS

The second chapter deals with the analyticity of solutions to the radial Schrodinger equation in the complex parameters λ and k and the results are applied to the Jost functions.

§2.1 Some Preliminary Lemmas

We begin by proving a lemma which is fundamental to the type of analysis with which we are dealing - namely Gronwall's Lemma - which may be stated as follows:

Lemma 2.1 (Gronwall)*

Let $f(x) \geq 0$, $g(x) \geq 0$ and $f(x)$ be measurable $f(x)g(x)$ and $g(x)$ integrable in $a \leq x \leq b$. If

(2.1) $$f(x) \leq C + \int_a^x f(t)g(t)dt \ , \quad C > 0 \ ,$$

then

(2.2) $$f(x) \leq C \exp(\int_a^x g(t)dt) \ .$$

Proof: Let $y = \int_a^x f(t)g(t)dt$ so that, for almost all x ,

$$\frac{dy}{dx} = f(x)g(x)$$

and

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* Titchmarsh, E. C., "Eigenfunction Expansions", Oxford, 1, p. 115(1962).

$$\frac{dy}{dx} \leq Cg(x) + yg(x) ,$$

$$\frac{d}{dx}\left[y \exp(-\int_a^x g(t)dt)\right] \leq Cg(x)\exp(-\int_a^x g(t)dt) .$$

Integration over (a,x) gives

$$y \exp(-\int_a^x g(t)dt) \leq C\left[1 - \exp(-\int_a^x g(t)dt)\right]$$

and

$$\begin{aligned} f(x) &\leq C + y \\ &\leq C \exp(\int_a^x g(t)dt) . \end{aligned}$$

An example of a use of Gronwall's Lemma is given in the proof of Lemma 2.2.

This thesis is essentially concerned with properties of solutions to second order linear ordinary differential equations

$$\frac{d^2y}{dx^2} + F(x)y = 0.$$

If $F(x)$ is an analytic function of x in a neighbourhood of $x = x_o$ with, at most, a second order pole at x_o then the differential equation may be solved by the standard power series methods. However, if F is not analytic then the usual approach is to try to find an equivalent integral equation for y which may be solved by some iterative process. We give here a method for constructing such integral equations which we believe to be new, and of which the well known variation of parameters results for Volterra and Fredholm-type integral equations are special cases.

Lemma 2.2

Let (a,b) be a finite or infinite real interval and F, F_1 and F_2 be real or complex-valued continuous functions on (a,b). If Y_1 and Y_2 satisfy the differential equations

$$\frac{d^2Y_1}{dx^2} + (F(x) + F_1(x))Y_1 = 0 \tag{2.3}$$

and

$$\frac{d^2Y_2}{dx^2} + (F(x) + F_2(x))Y_2 = 0 \tag{2.4}$$

respectively, and the Wronskian $W(Y_1(x), Y_2(x)) = Y_1 \frac{dY_2}{dx} - Y_2 \frac{dY_1}{dx} \neq 0$, $a < x < b$, then any solution of

$$\frac{d^2y}{dx^2} + F(x)y = 0 \tag{2.5}$$

satisfies

$$y(x) = f_{12}(x) + f_{21}(x) + \int_{x_1}^{x} g_{12}(x,t)F_1(t)y(t)dt + \int_{x_2}^{x} g_{21}(x,t)F_2(t)y(t)dt, \tag{2.6}$$

where x_i, $i = 1, 2$ are points of $[a,b]$ and

$$f_{ij}(x) = Y_j(x)W(Y_i(x_i), y(x_i))/W(Y_i(x), \; Y_j(x))$$

$$g_{ij}(x,t) = Y_j(x)Y_i(t)/W(Y_i(x), Y_j(x)), \quad i \neq j .$$

This result enables us to construct integral equations (2.6) which are equivalent to the differential equation (2.5) when the Wronskians of y and Y_i are given at $x = x_i$. We give as corollaries important

special cases which may be found in textbooks on differential equations and mathematical physics.

Corollary 1

If $F_1 = F_2 = F_o$, and $x_1 = x_2 = x_o$, then $W(Y_1, Y_2) = \Delta$, a constant, and y satisfies the equation (2.5) if, and only if,

$$(2.7) \qquad y(x) = f(x) + \int_{x_o}^{x} g(x,t)F_o(t)y(t)dt\ , \quad x_o \in [a,b]$$

where

$$f(x) = \frac{1}{\Delta} Y_2(x)W(Y_1(x_o), y(x_o)) - \frac{1}{\Delta} Y_1(x)W(Y_2(x_o), y(x_o))$$

and

$$g(x,t) = \frac{1}{\Delta} Y_2(x)Y_1(t) - \frac{1}{\Delta} Y_1(x)Y_2(t)\ .$$

Equation (2.7) is a Volterra-type integral equation and may be used when equation (2.5) is given with two initial conditions at $x = x_o$.

Corollary 2

If $F_1 = F_2 = F_o$, and $x_1 = a$, $x_2 = b$, and $W(Y_1, Y_2) = \Delta$ as before then y satisfies (2.5) if, and only if,

$$(2.8) \qquad y(x) = f(x) + \int_{a}^{b} g(x,t)F_o(t)y(t)dt$$

where

$$f(x) = \frac{1}{\Delta} Y_2(x)W(Y_1(a), y(a)) - \frac{1}{\Delta} Y_1(x)W(Y_2(b), y(b))$$

and

$$\begin{aligned} g(x,t) &= \frac{1}{\Delta} Y_2(x)Y_1(t)\ , \qquad a < t \le x \\ &= \frac{1}{\Delta} Y_1(x)Y_2(t)\ , \qquad x \le t < b \end{aligned}$$

The equation (2.8) is of Fredholm type and may be useful if boundary conditions are given for (2.5) at $x = a$ and $x = b$.

In the present work only equations of Volterra type will be used so that we will be interested in Lemma 2.2 when $x_1 = x_2 = x_o$.

<u>Proof of Lemma 2.2</u>

We first prove that, given (2.3) and (2.4), equation (2.5) implies equation (2.6).

Multiplication of (2.5) by Y_1 and of (2.3) by y and subtraction of the resulting expressions yields the result

$$\begin{aligned} Y_1 \frac{d^2y}{dx^2} - y \frac{d^2Y_1}{dx^2} &= \frac{d}{dx} W(Y_1, y) \\ &= F_1(x)Y_1(x)y(x) \, . \end{aligned} \tag{2.9}$$

Integration of (2.9) from x_1 to x gives

$$\begin{aligned} Y_1 \frac{dy}{dx} - y \frac{dY_1}{dx} &= W(Y_1(x), y(x)) \\ &= W(Y_1(x_1), y(x_1)) + \int_{x_1}^{x} F_1(t)Y_1(t)y(t)dt \, . \end{aligned} \tag{2.10}$$

Similarly,

$$Y_2 \frac{dy}{dx} - y \frac{dY_2}{dx} = W(Y_2(x_2), y(x_2)) + \int_{x_2}^{x} F_2(t)Y_2(t)y(t)dt \, . \tag{2.11}$$

Multiplication of (2.10) by Y_2 and of (2.11) by Y_1 and subtraction yields the integral equation (2.6).

The proof of the converse result that (2.6) implies (2.5) depends critically on the nature of $g_{12}(x,t)F_1(t)$ and $g_{21}(x,t)F_2(t)$

and the points x_1 and x_2. As an example of a use of Gronwall's lemma we will supply a proof in the case when $x_1 = x_2 = x_o$ and $|g_{12}(x,t)F_1(t) + g_{21}(x,t)F_2(t)| < g(t)$ where $g \in L(a,b)$. Differentiation of (2.6) twice yields after some rather lengthy but easy manipulation the equality

$$(2.12) \qquad E(x) = \int_{x_1}^{x} g_{12}(x,t)F_1(t)E(t)dt + \int_{x_2}^{x} g_{21}(x,t)F_2(t)E(t)dt$$

where $E(x) = \frac{d^2y}{dx^2} + F(x)y$. Equation (2.12) holds quite generally and, under the present assumption, reads

$$E(x) = \int_{x_o}^{x} (g_{12}(x,t)F_1(t) + g_{21}(x,t)F_2(t))E(t)dt ,$$

and hence

$$|E(x)| < \epsilon + \int_{x_o}^{x} g(t)\ |E(t)|dt ,$$

for every $\epsilon > 0$, and by Gronwall's lemma (lemma 2.1)

$$|E(x)| \leq \epsilon \exp\left(\int_{x_o}^{x} g(t)dt\right)$$

so that $E(x) = 0$, i.e. $\frac{d^2y}{dx^2} + F(x)y = 0$, which is (2.5).

In some important cases in our work it will not be possible to find a suitable Lebesque integrable function $g(t)$ independent of x so that the proof in the preceding paragraph will not hold; however, it is evident that whenever the integral equation (2.6) has convergent iterative solutions

$$y(x) = \sum_{n=0}^{\infty} y_n(x)$$

where

$$y_o(x) = f_{12}(x) + f_{21}(x)$$

$$y_n(x) = \int_{x_1}^{x} g_{12}(x,t)F_1(t)y_{n-1}(t)dt + \int_{x_2}^{x} g_{21}(x,t)F_2(t)y_{n-1}(t)dt, \qquad n = 1, 2, \ldots ,$$

then $E(x)$, from (2.12) is that solution of (2.6) for which $W(Y_1(x_1),y(x_1)) = 0$ and $W(Y_2(x_2),y(x_2)) = 0$, i.e. $y_o = 0$ and hence $y_n(x) = 0$, $n = 1, 2, \ldots$, giving $y(x) = E(x) = 0$, or $\frac{d^2y}{dx^2} + F(x)y = 0$, as before.

The next three lemmas deal with conditions under which Volterra integral equations have convergent iterative solutions. Lemma 2.3 is well known but lemmas 2.4 and 2.5 are rather specialised and are probably new.

Lemma 2.3

The Volterra integral equation

$$y(x) = f(x) + \int_a^x K(x,t)y(t)dt \tag{2.13}$$

where $f(x)$ and $K(x,t)$ are measurable functions of their variables and

$$|f(x)| \leq \eta(x) , \quad x \in (a,b) \tag{2.14}$$

$$|K(x,t)\eta t)| \leq \eta(x)g(t) , \quad t \in (a,x) \tag{2.15}$$

where $\eta(x)(\neq 0)$ is a measurable almost everywhere finite function and $g \in L(a,b)$, $G(x) = \int_a^x g(t)dt$, then (2.13) has a solution

$$y(x) = \sum_{n=0}^{\infty} y_n(x) \tag{2.16}$$

where

$$y_o(x) = f(x)$$

$$y_n(x) = \int_a^x K(x,t)y_{n-1}(t)dt \ , \qquad n = 1, 2, \dots ,$$

and, for almost all x ,

(2.17) $$|y_n(x)| \leq \eta(x) \frac{(G(x))^n}{n!} \ , \qquad n = 0, 1, 2, \dots ,$$

(2.18) $$|y(x)| \leq \eta(x) \exp(G(x))$$

(2.19) $$|y(x) - \sum_{n=0}^{N-1} y_n(x)| \leq \eta(x) \frac{(G(x))^n}{n!} \exp(G(x))$$

and in particular

(2.20) $$|y(x) - f(x)| \leq \eta(x)G(x) \exp(G(x)) \ .$$

Also $y(x)$ is the only solution to (2.13). This result is Theorem 1 of Erdélyi's work [24].

<u>Proof</u>

Inequality (2.17) will be proved by induction. The function $f(x)$ is measurable and (2.17) holds by hypothesis for $n = 0$. If $y_{n-1}(x)$ is measurable and

$$|y_{n-1}(x)| \leq \eta(x) \frac{(G(x))^{n-1}}{(n-1)!}$$

then

$$|y_n(x)| \leq \int_a^x |K(x,t)|\eta(t) \frac{(G(t))^{n-1}}{(n-1)!} dt$$

$$\leq \eta(x) \int_a^x g(t) \frac{(G(t))^{n-1}}{(n-1)!} dt$$

$$= \eta(x) \frac{(G(x))^n}{n!}$$

and (2.17) holds for all n and hence (2.18) is true. From (2.15), (2.18) and the Lebesgue dominated convergence theorem we may let $N \to \infty$ in the equation

$$\sum_{n=0}^{N} y_n(x) = f(x) + \int_a^x K(x,t) \sum_{n=0}^{N-1} y_n(t)dt$$

so that $y(x) = \sum_{n=0}^{\infty} y_n(x)$ is a solution of (2.13).

To prove (2.19) we use

$$\frac{1}{n!} \leq \frac{1}{N!(n-N)!}$$

so that

$$|y - \sum_{n=0}^{N-1} y_n| \leq \sum_{n=N}^{\infty} |y_n| \leq \sum_{n=N}^{\infty} \eta \frac{G^n}{n!} \leq \eta \frac{G^N}{N!} \sum_{n=N}^{\infty} \frac{G^{n-N}}{(n-N)!}$$

$$= \eta e^G \frac{G^N}{N!} .$$

The uniqueness may be proved by observing that if $z(x)$ is the difference of two solutions to (2.13) then

$$z(x) = \int_a^x K(x,t)z(t)dt$$

and if $Z(x) = z(x)/\eta(x)$ then

$$Z(x) = \int_a^x K(x,t) \frac{\eta(t)}{\eta(x)} Z(t)dt$$

and

$$|Z(x)| \leq \int_a^x g(t)|Z(t)|dt$$

$$\leq \epsilon + \int_a^x g(t)|Z(t)|dt \quad \text{for every} \quad \epsilon > 0$$

and hence from Gronwall's lemma $Z(x) = 0$, i.e. $z(x) = 0$.

In some cases when no suitable function $g \in L(a,b)$ exists we will need lemmas 2.4 and 2.5.

Lemma 2.4

If y satisfies a Volterra integral equation

$$y(x) = f(x) + \int_0^x K(x,t)y(t)dt \tag{2.21}$$

where $f(x)$ and $K(x,t)$ are measurable functions of their variables and $\eta(x)$ is measurable and finite almost everywhere such that

$$|f(x)| \leq \eta(x) , \tag{2.22}$$

and

$$|K(x,t)\eta(t)| \leq \eta(x)M(t^{\delta-1} + \frac{t^{2\mu-1}}{x^{2\mu}}) , \tag{2.23}$$

$\delta > 0$, $\mu > M$ and $\frac{1}{2\mu} > \frac{x^\delta}{\delta}$,

then

$$y(x) = \sum_{n=0}^{\infty} y_n(x) \tag{2.24}$$

where y_n are defined as in lemma 2.3 and

$$|y_n(x)| \leq \eta(x) \left(\frac{M}{\mu}\right)^n \tag{2.25}$$

$$|y(x)| \leq \eta(x) \frac{\mu}{\mu - M} . \tag{2.26}$$

The proof is very like that of lemma 2.3 and is outlined below.

Proof

Inequality (2.25) holds for $n = 0$ by hypothesis. If $|y_{n-1}(x)| \leq \eta(x) \left(\frac{M}{\mu}\right)^{n-1}$ then

$$
\begin{aligned}
|y_n(x)| &= \left|\int_0^x K(x,t)y_{n-1}(t)dt\right| \\
&\leq \left(\frac{M}{\mu}\right)^{n-1} \int_0^x |K(x,t)\eta(t)|dt \\
&\leq \eta(x) \left(\frac{M}{\mu}\right)^{n-1} M \int_0^x \left(t^{\delta-1} + \frac{t^{2\mu-1}}{x^{2\mu}}\right)dt \\
&= \eta(x) \left(\frac{M}{\mu}\right)^{n-1} M\left(\frac{x^\delta}{\delta} + \frac{1}{2\mu}\right) \\
&< \eta(x) \left(\frac{M}{\mu}\right)^{n}
\end{aligned}
$$

so that (2.25) holds for all n. The rest of the proof is the same as in lemma 2.3.

<u>Lemma 2.5</u>

The integral equation

(2.27) $$y(x) = f(x) + \int_x^\infty K(x,t)t(t)dt,$$

where f and K are measurable and

(2.28) $$|f(x)| \leq \eta(x)$$

(2.29) $$|K(x,t)\eta(t)| \leq \eta(x)\, M\left(\frac{1}{t^{1+\gamma}} + \exp(2\alpha(x-t))\right),$$

$\gamma > 0$, $\alpha > M$ and $2\alpha < \gamma x^\gamma$ and $\eta(x)$ is measurable and finite almost everywhere, has a solution

(2.30) $$y(x) = \sum_{n=0}^{\infty} y_n(x)$$, y_n as before, and

(2.31) $$|y_n(x)| \leq \left(\frac{M}{\alpha}\right)^n \eta(x)$$

(2.32) $$|y(x)| \leq \frac{\alpha}{\alpha - M}\, \eta(x)$$

The proof is identical with that of lemma 2.4.

§2.2 Solutions to the Radial Schrödinger Equations

We now use lemmas 2.2 and 2.3 to establish the existence of certain solutions to the radial Schrödinger equation.

$$\frac{d^2y}{dx^2} + (k^2 - \frac{\lambda^2-\frac{1}{4}}{x^2} - V(x))y = 0, \qquad 0 < x < \infty , \tag{2.33}$$

where λ and k are complex parameters and $V(x)$ is continuous such that

$$\int_0^{R>0} t|V(t)|dt < \infty \tag{2.34}$$

$$\int_R^{\infty} |V(t)|dt < \infty . \tag{2.35}$$

Theorem 2.1

The equation (2.33) has two linearly independent solutions $F^{(1)}(\lambda,k; x)$ and $F^{(2)}(\lambda,k; x)$ which have the asymptotic behaviour

$$F^{(1)}(x) = (\frac{2}{k\pi})^{\frac{1}{2}} e^{i(kx-\frac{1}{2}\lambda\pi-\pi/4)}(1 + O(\Sigma(x))), \ \operatorname{Im}k \geq 0 , \tag{2.36}$$

$$F^{(2)}(x) = (\frac{2}{k\pi})^{\frac{1}{2}} e^{-i(kx-\frac{1}{2}\lambda\pi-\pi/4)}(1 + O(\Sigma(x))), \operatorname{Im}k \leq 0 , \tag{2.37}$$

$$\frac{d}{dx} F^{(1)}(x) = i(\frac{2k}{\pi})^{\frac{1}{2}} e^{i(kx-\frac{1}{2}\lambda\pi-\pi/4)}(1 + O(\Sigma(x))), \ \operatorname{Im}k \geq 0 , \tag{2.38}$$

and

$$\frac{d}{dx} F^{(2)}(x) = -i(\frac{2k}{\pi})^{\frac{1}{2}} e^{-i(kx-\frac{1}{2}\lambda\pi-\pi/4)}(1 + O(\Sigma(x))), \ \operatorname{Im}k \leq 0, \tag{2.39}$$

as $x \to \infty$,

where

$$\Sigma(x) = \max \{\frac{1}{x} , \int_x^{\infty} |V(t)|dt\} .$$

Corollary

The free $(V = 0)$ Schrödinger equation has solutions

$$F_o^{(1)}(x) = x^{\frac{1}{2}} H_\lambda^{(1)}(kx) \quad \text{and} \quad F_o^{(2)}(x) = x^{\frac{1}{2}} H_\lambda^{(2)}(kx)$$

whose behaviour as $x \to \infty$ is given by (2.36) - (2.39). $H_\lambda^{(1,2)}$ are the Hankel functions.

Proof:

Let

$$F(x) = k^2 - \frac{\lambda^2 - 1/4}{x^2} - V(x)$$

and

$$F_1(x) = F_2(x) = \frac{\lambda^2 - 1/4}{x^2} + V(x)$$

in lemma 2.2. Equation (2.5) is then the radial Schrödinger equation and equations (2.3) and (2.4) are both

$$\frac{d^2Y}{dx^2} + k^2 Y = 0 .$$

We may take $Y_1(x) = e^{ikx}$ and $Y_2(x) = e^{-ikx}$ so that $W(Y_1, Y_2) = -2ik$ and if $x_1 = x_2 = \infty$ the integral equation (2.6) reads

$$y(x) = Ae^{ikx} + Be^{-ikx} + \frac{1}{k}\int_x^\infty \mathrm{Sin}\,[k(t-x)]\left(\frac{\lambda^2 - 1/4}{t^2} + V(t)\right)y(t)dt \tag{2.40}$$

where A and B are arbitrary constants. Particular cases are

$$\begin{aligned} F^{(1)}(x) = {} & \left(\frac{2}{k\pi}\right)^{\frac{1}{2}} e^{i(kx - \frac{1}{2}\lambda\pi - \pi/4)} \\ & + \frac{1}{k}\int_x^\infty \mathrm{Sin}\,[k(t-x)]\left(\frac{\lambda^2 - 1/4}{t^2} + V(t)\right)F^{(1)}(t)dt \end{aligned} \tag{2.41}$$

and

$$(2.42) \qquad F^{(2)}(x) = \left(\frac{2}{k\pi}\right)^{\frac{1}{2}} e^{-i(kx-\frac{1}{2}\lambda\pi-\pi/4)} + \int_x^\infty \operatorname{Sin}[k(t-x)]\left(\frac{\lambda^2-1/4}{t^2} + V(t)\right)F^{(2)}(t)dt$$

To apply lemma 2.3 to equation (2.41) we note that

$$f(x) = \left(\frac{2}{k\pi}\right)^{\frac{1}{2}} e^{i(kx-\frac{1}{2}\lambda\pi-\pi/4)}$$

$$K(x,t) = \frac{1}{k}\operatorname{Sin}[k(t-x)]\left(\frac{\lambda^2-1/4}{t^2} + V(t)\right).$$

We may take $\eta(x) = \left|\frac{2}{k\pi}\right|^{\frac{1}{2}} e^{-\operatorname{Im}(kx-\frac{1}{2}\lambda\pi)}$ so that

$$|K(x,t)\eta(t)| \le \eta(x)g(t)\,, \qquad \operatorname{Im}k \ge 0$$

where

$$g(t) = \frac{1}{|k|}\left|\frac{\lambda^2-1/4}{t^2} + V(t)\right|$$

and the results of lemma 2.3 apply when $\operatorname{Im}k \ge 0$, and $x \in (R, \infty)$, $R > 0$. In particular, from (2.20),

$$F^{(1)}(x) = \left(\frac{2}{k\pi}\right)^{\frac{1}{2}} e^{i(kx-\frac{1}{2}\lambda\pi-\pi/4)}(1 + O(\Sigma(x))), \quad \operatorname{Im}k \ge 0$$

and similarly

$$F^{(2)}(x) = \left(\frac{2}{k\pi}\right)^{\frac{1}{2}} e^{-i(kx-\frac{1}{2}\lambda\pi-\pi/4)}(1 + O(\Sigma(x))), \quad \operatorname{Im}k \le 0, \text{ as } x \to \infty.$$

From (2.41)

$$\frac{d}{dx}F^{(1)}(x) = i\left(\frac{2k}{\pi}\right)^{\frac{1}{2}} e^{i(kx-\frac{1}{2}\lambda\pi-\pi/4)} - \int_x^\infty \cos[k(t-x)]\left(\frac{\lambda^2-1/4}{t^2} + V(t)\right)F^{(1)}(t)dt$$

$$= i\left(\frac{2k}{\pi}\right)^{\frac{1}{2}} e^{i(kx-\frac{1}{2}\lambda\pi-\pi/4)}(1 + O(\Sigma(x))), \quad \operatorname{Im}k \ge 0$$

and from (2.42)

$$\frac{d}{dx}F^{(2)}(x) = -i\left(\frac{2k}{\pi}\right)^{\frac{1}{2}} e^{-i(kx-\frac{1}{2}\lambda\pi-\pi/4)}(1 + O(\Sigma(x))), \quad \mathrm{Im}k \leq 0$$

as $x \to \infty$.

That $F^{(1)}$ and $F^{(2)}$ are linearly independent $(k \neq 0)$ can be seen from $W(F^{(1)}, F^{(2)}) = \frac{4}{i\pi} \neq 0$. The only values of k for which we have established the existence of both solutions are $\mathrm{Im}k = 0$.

Theorem 2.2

The radial Schrödinger equation (2.33) has two linearly independent solutions $\varphi(\lambda,k;\ x)$ and $\varphi(-\lambda,k;\ x)$ with the asymptotic behaviour

(2.43) $$\varphi(\lambda,k;\ x) = \left(\frac{k}{2}\right)^{\lambda}(\Gamma(\lambda+1))^{-1} x^{\lambda+\frac{1}{2}}(1 + O(\delta(x))), \quad \mathrm{Re}\lambda \geq 0,$$

and

(2.44) $$\frac{d}{dx}\varphi(\lambda,k;\ x) = (\lambda+\tfrac{1}{2})\left(\frac{k}{2}\right)^{\lambda}(\Gamma(\lambda+1))^{-1} x^{\lambda-\frac{1}{2}}(1 + O(\delta(x))), \quad \mathrm{Re}\lambda \geq 0,$$

as $x \to 0$,

where

$$\delta(x) = \max\{x^2, \int_0^x t|V(t)|dt\}.$$

Corollary

The free $(V = 0)$ Schrödinger equation has two linearly independent solutions $\varphi_o(\pm\lambda,k;\ x) = x^{\frac{1}{2}} J_{\pm\lambda}(kx)$ $(\lambda \neq 0, \pm1, \pm2, \ldots)$, whose behaviour as $x \to 0$ is given by (2.43) and (2.44).

Proof:

In lemma 2.2 we take

$$F(x) = k^2 - \frac{\lambda^2-1/4}{x^2} - V(x)$$

and

$$F_1(x) = F_2(x) = V(x) - k^2$$

so that (2.3) and (2.4) are the same equation

$$\frac{d^2Y}{dx^2} - \frac{\lambda^2-1/4}{x^2} Y = 0$$

and we may take $Y_1(x) = x^{\lambda+\frac{1}{2}}$, $Y_2(x) = x^{-\lambda+\frac{1}{2}}$ and $x_1 = x_2 = 0$. We now have $W(Y_1, Y_2) = -2\lambda$ and the radial Schrödinger equation is equivalent to the integral equation

(2.45) $$y(x) = Ax^{\lambda+\frac{1}{2}} + Bx^{-\lambda+\frac{1}{2}} + \frac{1}{2\lambda}\int_0^x \left(\frac{t^{\lambda+\frac{1}{2}}}{x^{\lambda-\frac{1}{2}}} - \frac{x^{\lambda+\frac{1}{2}}}{t^{\lambda-\frac{1}{2}}}\right)(k^2-V(t))y(t)dt.$$

A particular solution is given by

(2.46) $$\varphi(\lambda,k;\, x) = \left(\frac{k}{2}\right)^{\lambda}(\Gamma(\lambda+1))^{-1} x^{\lambda+\frac{1}{2}} + \frac{1}{2\lambda}\int_0^x \left(\frac{t^{\lambda+\frac{1}{2}}}{x^{\lambda-\frac{1}{2}}} - \frac{x^{\lambda+\frac{1}{2}}}{t^{\lambda-\frac{1}{2}}}\right)(k^2-V(t))\varphi(\lambda,k;\, t)dt\,.$$

Here

$$f(x) = \left(\frac{k}{2}\right)^{\lambda}(\Gamma(\lambda+1))^{-1} x^{\lambda+\frac{1}{2}}$$

and

$$K(x,t) = \frac{1}{2\lambda}\left(\frac{t^{\lambda+\frac{1}{2}}}{x^{\lambda-\frac{1}{2}}} - \frac{x^{\lambda+\frac{1}{2}}}{t^{\lambda-\frac{1}{2}}}\right).$$

Let $\eta(x) = \left|\left(\frac{k}{2}\right)^{\lambda}(\Gamma(\lambda+1))^{-1}\right| x^{\mathrm{Re}\lambda+\frac{1}{2}}$ so that

$$|K(x,t)\eta(t)| \leq \eta(x)g(t)\,, \quad \mathrm{Re}\lambda \geq 0$$

where $g(t) = \frac{t}{|\lambda|}\,|k^2 - V(t)|$ and lemma 2.3 holds on any interval $(0,R)$, $R < \infty$. The rest of the proof is the same as in Theorem 2.1.

The existence of both solutions has been established only on the line $\mathrm{Re}\lambda = 0$.

§2.3 Jost Functions

We now give a more precise definition of the Jost functions $f^{(1,2)}(\lambda,k)$ than that of Chapter 1. In Chapter I we were concerned with a solution $\varphi_\ell(k;\ x)$ to the radial Schrödinger equation which is bounded and has a continuous bounded derivative when ℓ is a non-negative integer. The function φ_ℓ must therefore satisfy the integral equation (2.45), $\lambda = \ell + \frac{1}{2}$, and consideration of the behaviour of solutions to this integral equation as $x \to 0$ shows that we must have $B = 0$ and $A \neq 0$ in this case. Therefore we may take $\varphi_\ell(k;\ x) = \varphi(\lambda,k;\ x)$, $\lambda = \ell + \frac{1}{2}$, and we define the Jost functions and their continuations to complex λ and k by the relation

$$(2.47) \qquad \varphi(\lambda,k;\ x) = \tfrac{1}{2}(f^{(1)}(\lambda,k)F^{(2)}(\lambda,k;\ x) + f^{(2)}(\lambda,k)F^{(1)}(\lambda,k;\ x))$$

where $F^{(1,2)}(x)$ are the linearly independent solutions to (2.33) introduced in Theorem 2.1. From the asymptotic behaviour of $F^{(1,2)}(x)$ as $x \to \infty$ we notice that this is the same definition as that given by (1.9) and (1.10). The free $(V = 0)$ Jost functions are given by the relation

$$\varphi_0(\lambda,k;\ x) = \tfrac{1}{2}(F_0^{(1)}(\lambda,k;\ x) + F_0^{(2)}(\lambda,k;\ x))$$

i.e.

$$x^{\frac{1}{2}}J_\lambda(kx) = \tfrac{1}{2}(x^{\frac{1}{2}}H_\lambda^{(1)}(kx) + x^{\frac{1}{2}}H_\lambda^{(2)}(kx))$$

so that

$$f_0^{(1)}(\lambda,k) = f_0^{(2)}(\lambda,k) = 1\ .$$

Theorems 2.1 and 2.2 and their corollaries and the definition (2.47) allow us to derive a number of expressions for the Jost functions:

$$(2.48)\qquad f^{(1)}(\lambda,k) = \frac{\pi}{2i} W(\varphi(x), F_o^{(1)}(x))_{x=\infty}, \quad \operatorname{Im}k \geq 0 ,$$

$$= \frac{\pi}{2i} W(\varphi(x), F^{(1)}(x)), \quad 0 < x < \infty ,$$

$$= \frac{\pi}{2i} W(\varphi_o(x), F^{(1)}(x))_{x=0}, \quad \operatorname{Re}\lambda \geq 0 .$$

Similarly,

$$(2.49)\qquad f^{(2)}(\lambda,k) = \frac{i\pi}{2} W(\varphi(x), F_o^{(2)}(x))_{x=\infty}, \quad \operatorname{Im}k \leq 0 ,$$

$$= \frac{i\pi}{2} W(\varphi(x), F^{(2)}(x)), \quad 0 < x < \infty$$

$$= \frac{i\pi}{2} W(\varphi_o(x), F^{(2)}(x))_{x=0}, \quad \operatorname{Re}\lambda \geq 0 .$$

<u>Theorem 2.3</u>

The Jost functions have the integral representations

$$(2.50)\qquad f^{(1,2)}(\lambda,k) = 1 \pm \frac{i\pi}{2} \int_o^\infty x^{\frac{1}{2}} V(x) H_\lambda^{(1,2)}(kx) \varphi(\lambda,k;\ x)dx$$

$$\operatorname{Re}\lambda \geq 0, \quad \operatorname{Im}k \geq 0, \quad \operatorname{Im}k \leq 0$$

$$(2.51)\qquad f^{(1,2)}(\lambda,k) = 1 \pm \frac{i\pi}{2} \int_o^\infty x^{\frac{1}{2}} V(x) J_\lambda(kx) F^{(1,2)}(\lambda,k;\ x)dx$$

$$\operatorname{Re}\lambda \geq 0, \quad \operatorname{Im}k \geq 0, \quad \operatorname{Im}k \leq 0.$$

<u>Corollary</u>

From the Weber-Graf circuit relations for the Hankel functions and (2.50) we have

$$(2.52)\qquad f^{(1)}(\lambda,ke^{i\pi}) = e^{-\lambda\pi i} f^{(2)}(\lambda,k), \quad -\pi \leq \arg k \leq 0$$

and

$$(2.53) \qquad f^{(2)}(\lambda, ke^{i\pi}) = 2\cos\lambda\pi\, f^{(2)}(\lambda,k) - e^{\lambda\pi i} f^{(1)}(\lambda,k), \quad k \text{ real} > 0$$

since $\varphi(\lambda, ke^{i\pi}; x) = \varphi(\lambda,k; x)\, e^{i\pi\lambda}$.

Proof:

$$\text{Let} \quad F(x) = k^2 - \frac{\lambda^2 - 1/4}{x^2} - V(x)$$

$$F_1(x) = V(x)$$

in lemma 2.2. We may then take $y(x) = \varphi(\lambda,k; x)$ and $Y_1(x) = F_o^{(1)}(\lambda,k; x) = x^{\frac{1}{2}} H_\lambda^{(1)}(kx)$ and equation (2.9) then reads

$$\frac{d}{dx} W(\varphi(x), F_o^{(1)}(x)) = -V(x) F_o^{(1)}(x)\varphi(x) .$$

Integration of this expression from 0 to ∞ and (2.48) give the expression (2.50) for $f^{(1)}(\lambda,k)$, $\mathrm{Re}\lambda \geq 0$, $\mathrm{Im}k \geq 0$. The other results in Theorem 2.3 are proved in the same way.

§2.4 Analyticity of $F^{(1,2)}(\lambda,k; x)$ and $\varphi(\lambda,k; x)$

By direct differentiation with respect to x of the integral equations (2.41), (2.42) and (2.46) we readily see that $F^{(1,2)}(\lambda,k; x)$ and $\varphi(\lambda,k; x)$ each has the same regions of analyticity as its derivative in complex λ for fixed k and x and in complex k for fixed λ and x and hence, from the formulae

$$f^{(1,2)}(\lambda,k) = \pm \frac{\pi}{2i} W(\varphi(\lambda,k; x), F^{(1,2)}(\lambda,k; x))$$

we see that these regions are also the regions of analyticity of the Jost functions.

<u>Theorem 2.4</u>

For fixed λ and x $(x > 0)$ the functions $F^{(1,2)}(\lambda,k;\ x)$ are regular analytic functions of k in $\mathrm{Im}k \geq 0$ and $\mathrm{Im}k \leq 0$ respectively. If k and x are fixed these functions are entire in λ.

<u>Proof:</u>

From the integral equation (2.41) for $F^{(1)}$ and lemma 2.3 we see that

$$F^{(1)}(\lambda,k;\ x) = \sum_{n=0}^{\infty} F_n^{(1)}(\lambda,k;\ x)$$

where the $F_n^{(1)}$ are regular analytic functions in $\mathrm{Im}k \geq 0$ and entire functions of λ and

$$|F_n^{(1)}| \leq \eta \frac{(G)^n}{n!}, \quad \mathrm{Im}k \geq 0$$

where

$$\eta(x) = \left|\frac{2}{k\pi}\right|^{\frac{1}{2}} e^{-\mathrm{Im}(kx-\frac{1}{2}\lambda\pi)}$$

and

$$G(x) = \frac{1}{|k|} \int_x^{\infty} \left|\frac{\lambda^2-1/4}{t^2} + V(t)\right| dt$$

so that $\sum F_n^{(1)}$ converges uniformly with respect to λ and k, $k \neq 0$, $\mathrm{Im}k \geq 0$ and hence the theorem for $F^{(1)}$ holds.

The theorem for $F^{(2)}$ can be proved in an identical manner.

We have now shown that $F^{(1,2)}(\lambda,k;\ x)$ are entire functions of λ and regular analytic functions of k in $0 \leq \arg k \leq \pi$ and $-\pi \leq \arg k \leq 0$ respectively for any continuous potential V such that $\int_{R \geq 0}^{\infty} |V(t)|dt < \infty$.

Froissart [27] and Bottino et al [12] have extended the regions of analyticity in the k plane under extra restrictions on the potential. We give here an independent proof of a theorem which essentially covers their results.

Theorem 2.5

If x is considered to be a complex variable and $V(x)$ is a regular analytic function in a wedge $-\theta_1 \leq \arg x \leq \theta_2$ $(\theta_{1,2} > 0)$ and if $\int_{x_o>0}^{\infty e^{i\alpha}} |V(t)|dt$ is uniformly bounded with respect to α then $F^{(1,2)}(\lambda,k;\, x)$ are regular analytic functions in $0 \leq \arg kx \leq \pi$ and $-\pi \leq \arg kx \leq 0$ respectively.

Proof:

Define a function $F_*^{(1)}(\lambda,k;\, x)$ on the wedge by

$$F_*^{(1)}(\lambda,k;\, x) = \sum_{n=0}^{\infty} F_{*n}^{(1)}(\lambda,k;\, x) \tag{2.54}$$

where

$$F_{*o}^{(1)}(\lambda,k;\, x) = \left(\frac{2}{k\pi}\right)^{\frac{1}{2}} e^{i(kx-\frac{1}{2}\lambda\pi-\pi/4)} \tag{2.55}$$

and

$$F_{*n}^{(1)}(\lambda,k;\, x) = \frac{1}{k}\int_x^{\infty e^{i\arg x}} \mathrm{Sin}[k(t-x)]\left(\frac{\lambda^2-1/4}{t^2} + V(t)\right)F_{*n-1}^{(1)}(\lambda,k;t)dt. \tag{2.56}$$

Each $F_{*n}^{(1)}$ is an entire function of λ and a regular analytic function of k in $0 \leq \arg kx \leq \pi$, and by induction

$$|F_{*n}^{(1)}| \leq \eta\,\frac{G^n}{n!}\,,\ 0 \leq \arg kx \leq \pi,\quad n = 0, 1, 2, \ldots ,$$

so that

$$|F_*^{(1)}| \leq \eta \exp(G), \quad 0 \leq \arg kx \leq \pi ,$$

where

$$\eta = \left|\frac{2}{k\pi}\right|^{\frac{1}{2}} e^{-\mathrm{Im}(kx-\frac{1}{2}\lambda\pi)}$$

and

$$G(x) = \frac{1}{|k|} \int_x^{\infty e^{i\arg x}} \left|\frac{\lambda^2-1/4}{t^2} + V(t)\right| |dt|.$$

The function $F_*^{(1)}(\lambda,k;\ x)$ is thus a regular analytic function of k (fixed λ and x) in $0 \leq \arg kx \leq \pi$. When x is real $F_{*n}^{(1)} = F_n^{(1)}$ and $F_*^{(1)} = F^{(1)}$. We will now show that $F_*^{(1)}$ is a regular analytic function of x (fixed λ and k) in the wedge and thus gives the analytic continuation of $F^{(1)}$ in x .

By definition $F_{*o}^{(1)}(\lambda,k;\ x)$ is a regular function of x in the wedge $-\theta_1 \leq \arg x \leq \theta_2$. If $F_{*n-1}^{(1)}(\lambda,k;\ x)$ is regular in the wedge then $F_{*n}^{(1)}(\lambda,k;\ x)$ is also a regular function of x there if the integral in (2.56) is independent of the way the contour goes to infinity inside the wedge.

Consider $\frac{1}{k} \int_{C(x,R)} \mathrm{Sin}[k(t-x)](\frac{\lambda^2-1/4}{t^2} + V(t))F_{*n-1}^{(1)}(\lambda,k;\ t)dt$

where $C(x,R)$ is given in Figure 1. Under the stated conditions on the potential and since

$$|F_{*n-1}^{(1)}| \leq \eta \frac{G^{n-1}}{(n-1)!}$$

we have $\int_R^{|R|e^{i\arg x}} \rightarrow 0$ as $R \rightarrow \infty$ and hence, by Cauchy's theorem,

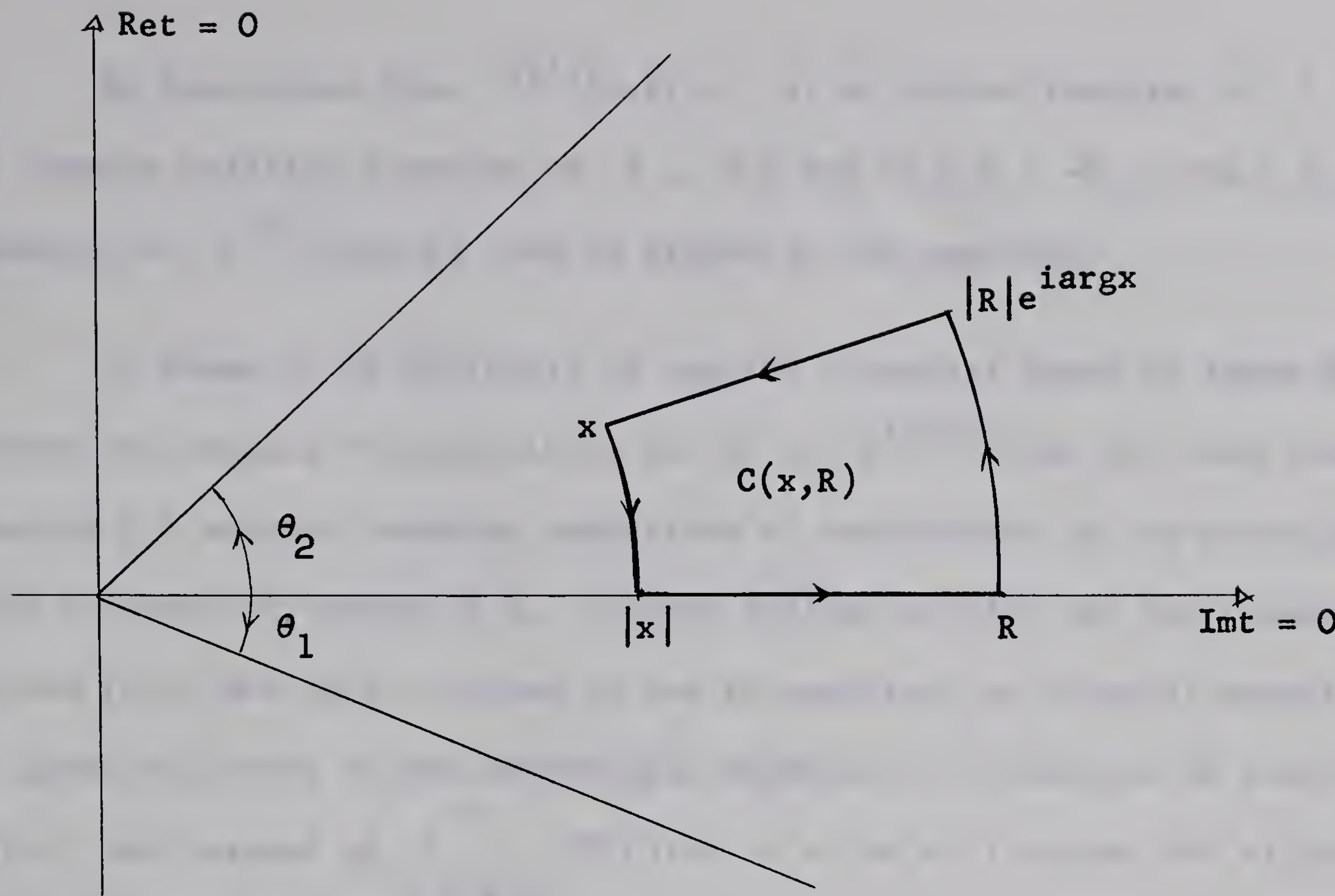

Figure 1. The Contour $C(x,R)$ in the t-plane.

$\int_x^{\infty e^{iargx}} = \int_{C(x)}$ if $F^{(1)}_{*n-1}$ is a regular function of x in the wedge

so that

$$F^{(1)}_{*n}(\lambda,k;\ x) = \frac{1}{k}\int_{C(x)} \mathrm{Sin}[k(t-x)]\left(\frac{\lambda^2-1/4}{t^2} + V(t)F^{(1)}_{*n-1}(\lambda,k;\ t)dt\right.$$

where $C(x)$ is the contour $(x,\ |x|,\ \infty)$ or any contour which joins x to ∞. $F^{(1)}_{*n}$ is thus a regular analytic function of x, $-\theta_1 \leq \arg x \leq \theta_2$, $n = 0, 1, 2, \ldots$, which implies that $F^{(1)}_{*}$ is regular there since $\sum F^{(1)}_{*n}$ converges uniformly with respect to x. $F^{(1)}_{*}(\lambda,k;\ x)$ is thus the analytic continuation of $F^{(1)}(\lambda,k;x)$ in $-\theta_1 \leq \arg x \leq \theta_2$ since $F^{(1)}_{*} - F^{(1)} = 0$ on the real line.

We have shown that $F^{(1)}(\lambda,k;\ x)$ is an entire function of λ and a regular analytic function of k, $0 \leq \arg kx \leq \pi$, $-\theta_1 \leq \arg x \leq \theta_2$. The result for $F^{(2)}(\lambda,k;\ x)$ can be proved in the same way.

It seems to be difficult to use the classical cases of lemma 2.2 to extend the regions of analyticity in k of $F^{(1,2)}(\lambda,k;\ x)$ from those of Theorem 2.4 without assuming conditions of analyticity on the potential similar to those of Theorem 2.5. In what follows we will use two 'comparison' equations (2.3) and (2.4) instead of one to construct an integral equation which gives solutions to the Schrödinger equation. In addition to continuity of $V(x)$ and instead of $\int_{R \geq 0}^{\infty} |V(t)|dt < \infty$ we will assume the slightly stronger condition

$$|V(x)| < \frac{C}{x^{1+\gamma}}\ , \quad \gamma > 0 \quad \text{for} \quad x > R \geq 0\ .$$

Before proving the next theorem we note that the differential equation

$$\frac{d^2Y}{dx^2} + (k^2 + \beta^2 e^{-2\alpha x})Y = 0$$

has a pair of solutions $J_{\pm \frac{ik}{\alpha}}(\frac{\beta}{\alpha} e^{-\alpha x})$ where J is a Bessel function. This can be proved by making the obvious substitutions in Bessel's equation.

<u>Theorem 2.6</u>

For any continuous potential $V(x)$ such that $|V(x)| < \frac{C}{x^{1+\gamma}}$, $\gamma > 0$, for $x > R \geq 0$, the functions $F^{(1,2)}(\lambda,k;\ x)$ are regular analytic functions throughout the k-plane cut from 0 to ∞ with the possible exception of singular points in $-\pi \leq \arg k \leq 0$ and $0 \leq \arg k \leq \pi$ respectively.

Proof:

That no singular points of $F^{(1,2)}$ can occur in $0 \leq \arg k \leq \pi$ and $-\pi \leq \arg \leq 0$ respectively is Theorem 2.4. To prove analyticity in the rest of the k-plane let

$$F(x) = k^2 - \frac{\lambda^2-1/4}{x^2} - V(x)$$

$$F_1(x) = \frac{\lambda^2-1/4}{x^2} + V(x)$$

and

$$F_2(x) = \frac{\lambda^2-1/4}{x^2} + V(x) + k^2 e^{-2\alpha x}$$

in lemma 2.2. We may then take

$$Y_1(x) = e^{-ikx}$$

and

$$Y_2(x) = \Gamma(\frac{ik}{\alpha} + 1)(\frac{k}{2\alpha})^{\frac{-ik}{\alpha}} J_{\frac{ik}{\alpha}}(\frac{k}{\alpha} e^{-\alpha x}) .$$

An integral equation which gives a solution to the Schrödinger equation is then given by (2.6) with $x_1 = x_2 = \infty$

$$y(x) = f(x) - \int_x^\infty K(x,t)y(t)dt \tag{2.57}$$

where

$$f(x) = \frac{Y_2(x)[W(Y_1(x),y(x))_{x=\infty}] - Y_1(x)[W(Y_2(x),y(x))_{x=\infty}]}{W(Y_1(x),\ Y_2(x))}$$

and

$$K(x,t) = \frac{Y_2(x)Y_1(t)F_1(t) - Y_1(x)Y_2(t)F_2(t)}{W(Y_1(x),\ Y_2(x))} .$$

If $\alpha > 0$ then, as $x \to \infty$,

$$Y_2(x) = e^{-ikx}(1 + O(e^{-2\alpha x}))$$

and

$$\frac{d}{dx} Y_2(x) = -ike^{-ikx}(1 + O(e^{-2\alpha x}))$$

so that $W(Y_1(x), Y_2(x))_{x=\infty} = 0$. Thus we must have $W(Y_1(x), y(x))_{x=\infty}$ $= W(Y_2(x), y(x))_{x=\infty}$ so that specification of these two Wronskians merely gives one boundary condition on y . We therefore see that equation (2.57) only gives two solutions (to within a multiplie) to the Schrődinger equation; one when $W(Y_{1,2}(x), y(x))_{x=\infty} = 0$, i.e. the trivial solution $y = 0$ and the other when $W(Y_{1,2}(x), y(x))_{x=\infty} = c \neq 0$. We will investigate the latter case and take $W(Y_{1,2}(x), y(x))_{x=\infty} = 1$ so that

$$f(x) = \frac{Y_2(x) - Y_1(x)}{W(Y_1(x), Y_2(x))}$$

$$= e^{ikx} \frac{\sum_{n=1}^{\infty} (-1)^n a_n e^{-2n\alpha x}}{\sum_{n=1}^{\infty} (-1)^{n+1} 2\alpha n a_n e^{-2n\alpha x}}$$

$$= -\frac{1}{2\alpha} e^{ikx}(1 + O(e^{-2\alpha x})), \quad \text{as} \quad x \to \infty \quad \text{and}$$

$K(x,t)$

$$= e^{ik(x,t)} \frac{\sum_{n=1}^{\infty} (-1)^n a_n (e^{-2n\alpha x} - e^{-2n\alpha t})(\frac{\lambda^2 - 1/4}{t^2} + V(t)) + \sum_{n=0}^{\infty} (-1)^{n+1} a_n k^2 e^{-(2n+2)dt}}{\sum_{n=1}^{\infty} (-1)^{n+1} 2\alpha n a_n e^{-2n\alpha x}}$$

$$= \frac{1}{2\alpha} e^{ik(x-t)} \Big\{ (e^{2\alpha(x-t)} - 1)(\frac{\lambda^2 - 1/4}{t^2} + V(t))(1 + O(e^{-2\alpha x}))$$

$$- \frac{k^2 \alpha}{ik + \alpha} e^{2\alpha(x-t)}(1 + O(e^{-2\alpha x})) \Big\} , \quad \text{as} \quad x \to \infty ,$$

where $a_n = (\frac{k}{2\alpha})^{2n} \frac{1}{n!\Gamma(\frac{ik}{\alpha} + n + 1)}$. The functions f and K are analytic in k and if we restrict ourselves away from singularities the bounds implied by the Hardy O's are uniform in α and k. We may then write

$$|f(x)| \leq \eta(x)$$

and

$$|K(x,t)\eta(t)| \leq \eta(x)g(t)$$

uniformly with respect to α and k where

$$\eta(x) = Ne^{-(\mathrm{Im}k)x}$$

and

$$g(t) = M\left\{\frac{1}{t^{1+\gamma}} + e^{2\alpha(x-t)}\right\}, \quad \gamma > 0 .$$

If we choose $\alpha > M$ and take x large enough to ensure $\gamma x^{\gamma} > 2\alpha$ in order that lemma 2.5 holds and

$$y(x) = \sum_{n=0}^{\infty} y_n(x)$$

where

$$|y_n(x)| \leq (\frac{M}{\alpha})^n \eta(x)$$

and

$$|(y(x)| \leq \frac{\alpha}{\alpha - M} \eta(x) .$$

Each y_n is an analytic function of k throughout the k-plane cut as described and $\sum_{n=0}^{\infty} y_n$ converges uniformly with respect to k in regions of regularity so that $y(x)$ is an analytic function of k when $\gamma x^{\gamma} > 2\alpha$ and k is restricted away from singularities of $f(x)$ and $\int_x^{\infty} K(x,t) \frac{f(t)}{f(x)} dt$.

We therefore have the result that $y(x)$ is an analytic function of k throughout the cut k-plane if $\gamma x^{\gamma} > 2\alpha$ and there are possible singularities when $f(x)$ or $\int_x^\infty K(x,t)\frac{f(t)}{f(x)}\,dt$ is singular.

To identify the solution $y(x)$ which may be written

$$y(x) = c_1(\lambda,k)F^{(1)}(\lambda,k;\, x) + c_2(\lambda,k)F^{(2)}(\lambda,k;\, x)$$

we consider its behaviour as $x \to \infty$. From (2.57) and lemma 2.5

$$y(x) = -\frac{1}{2\alpha}\, e^{ikx}(1 + \epsilon(x)), \quad \text{when } 0 \le \arg k \le \pi \text{ , say,}$$

where $|\epsilon(x)| \le \frac{M}{\alpha}(\frac{\alpha}{\alpha-M})$, $x > (\frac{2\alpha}{\gamma})^{1/\gamma}$, so that from Theorem 2.1 ,

$$c_1(\lambda,k) = -\frac{1}{2\alpha}\left(\frac{k\pi}{2}\right)^{1/2} e^{i(\frac{1}{2}\lambda\pi + \pi/4)}$$

and

$$c_2(\lambda,k) = 0 \text{ .}$$

Thus we have proved the theorem for $F^{(1)}(\lambda,k;\, x)$ when $\gamma x^{\gamma} > 2\alpha$. To extend the result to $0 < x < \infty$ we note that

$$F^{(1)}(x) = g(x) + \int_x^{x_o} H(x,t)F^{(1)}(t)dt$$

where

$$g(x) = \frac{1}{2ik}\,\{e^{-ikx}W(F^{(1)}(x_o),e^{ikx_o}) - e^{ikx}W(F^{(1)}(x_o),e^{-ikx_o})\}$$

and

$$H(x,t) = \frac{1}{k}\mathrm{Sin}[k(t-x)](\frac{\lambda^2-1/4}{t^2} + V(t)) \text{ .}$$

If $x_o < \infty$ the iterated solution to this integral equation converges uniformly with respect to k in any region in the k-plane which is free

of singularities of $g(x)$. Thus $F^{(1)}(x)$ has the same singularities in the k-plane as $F^{(1)}(x_o)$, $x_o \geq x$. We have shown that the singularities of $F^{(1)}(\lambda,k;\, x)$ in the k-plane are independent of x so that without loss of generality we may say that the only possible singularities of $F^{(1)}(\lambda,k;\, x)$ are those of $f(x_o)$ and $\int_{x_o}^{\infty} K(x_o,t)\, \frac{f(t)}{f(x_o)}\, dt$. However, the singularities of $f(x_o)$ are functions of α which is arbitrary so that the integral gives the only possible singularities. It should be noted that the singular points are not necessarily isolated.

The theorem for $F^{(2)}(\lambda,k;\, x)$ can be proved by taking $Y_1(x) = e^{ikx}$ and $Y_2(x) = \Gamma(\frac{-ik}{\alpha} + 1)(\frac{k}{\alpha})^{\frac{-ik}{\alpha}} J_{\frac{-ik}{\alpha}}(\frac{k}{\alpha}\, e^{-\alpha x})$ and proceeding as before.

We now turn our attention to the analyticity of $\varphi(\lambda,k;\, x)$. Since the theorems on the analyticity in λ of φ are very similar to the theorems on analyticity in k of $F^{(1,2)}$ we will omit most of the details of the proofs.

<u>Theorem 2.7</u>

The $\varphi(\lambda,k;\, x)$ is regular analytic in $\mathrm{Re}\lambda \geq 0$ (fixed k, x) and $(\frac{2}{k})^{\lambda}\, \varphi(\lambda,k;\, x)$ is an entire function of k (fixed x, λ) for any continuous potential V such that $\int_0^{R>0} t\,|V(t)|\,dt < \infty$.

This theorem can be proved using the integral equation (2.46) with $\eta(x)$ and $g(t)$ as in Theorem 2.2 so that

$$\varphi = \sum_{n=0}^{\infty} \varphi_n$$

where each φ_n is regular analytic in $\text{Re}\lambda \geq 0$ and $(\frac{2}{k})^{\lambda} \varphi_n$ is entire in k and

$$|\varphi_n(x)| \leq \eta(x) \frac{(G(x))^n}{n!} \quad , \quad \text{Re}\lambda \geq 0 \ ,$$

where

$$\eta(x) = |(\tfrac{k}{2})^{\lambda} (\Gamma(\lambda+1))^{-1}| \, x^{\text{Re}\lambda+\frac{1}{2}}$$

and

$$G(x) = \frac{1}{|\lambda|} \int_0^x t|k^2 - V(t)| \, dt$$

giving the uniform convergence of the series.

The continuation of $\varphi(\lambda,k; x)$ to $\text{Re}\lambda \leq 0$ is of the same order of difficulty as the continuation of $F^{(1,2)}(\lambda,k; x)$ to $\text{Im}k \leq 0$ and $\text{Im}k \geq 0$ respectively. The problem has been solved when, in addition to $\int_0^R t \, |V(t)|dt < \infty$, $V(x)$ is an analytic function of complex x in a neighbourhood of $x = 0$ by Froissart [27] and special cases have been considered by Challifour and Eden [18], Cheng [19], Mandelstam [32] and Squires [42] and many others. The only study of non-analytic potentials before ours seems to be that of Newton [35] who showed that if $xV(x)$ is m times differentiable at $x = 0$ then $\varphi(\lambda,k; x)$ is regular analytic in $\text{Re}\lambda > -\frac{(m+1)}{2}$ except for possible poles at $\lambda = -\frac{n}{2}$, $n = 1, 2, \ldots$.

Before presenting our results for non-analytic potentials we introduce a contour integral technique like that of Theorem 2.5 for analytic $V(x)$ with possible branch points at $x = 0$ such that $V(xe^{i\alpha\pi}) = V(x)$ where α is some positive real number. We have been able to develop this technique to deal with potentials $V(x) = \sum_n v_n(x)$

where $v_n(xe^{i\alpha_n\pi}) = v_n(x)$ and potentials with logarithmic branch points at the origin, i.e. $v_n(xe^{i\alpha_n\pi}) = v_n(x) + c_n$.

We give only the simple case to give some idea of the method.

Theorem 2.8

If $V(x)$ is an analytic function of the complex variable x near $x = 0$ and has a branch point at $x = 0$ such that $V(xe^{i\alpha\pi}) = V(x)$ and also $\int_0^x |tV(t)|\,|dt|$ is bounded uniformly for all x , $0 \leq \arg x \leq \alpha\pi$, in a closed neighbourhood of $x = 0$, the integral being along the line segment $(0,x)$, then $\varphi(\lambda,k;\, x)$ is an analytic function of λ throughout the λ-plane with possible singularities at $\lambda = \frac{2m}{\alpha} - s - \frac{1}{2}$, $s = 1, 2, \ldots$, where m is an integer such that $\frac{2m}{\alpha} < s + \frac{1}{2}$.

Proof:

The analytic continuation of φ to complex x can be given by $\varphi = \sum_{n=0}^{\infty} \varphi_n$ where

$$\varphi_0(x) = \left(\frac{k}{2}\right)^{\lambda} (\Gamma(\lambda+1))^{-1}\, x^{\lambda+\frac{1}{2}} ,$$

and

$$\varphi_n(x) = \int_0^x K(x,t)\varphi_{n-1}(t)dt, \qquad \mathrm{Re}\lambda \geq 0,\ n = 1, 2, \ldots ,$$

the integral being along the line segment $(0,x)$ and

$$K(x,t) = \frac{1}{2\lambda}\left(\frac{t^{\lambda+\frac{1}{2}}}{x^{\lambda-\frac{1}{2}}} - \frac{x^{\lambda+\frac{1}{2}}}{t^{\lambda-\frac{1}{2}}}\right) (k^2 - V(t)) .$$

By induction on n

$$\varphi_n(xe^{i\alpha\pi}) = e^{i\alpha\pi(\lambda+n+\frac{1}{2})}\varphi_n(x)$$

Hence by Cauchy's Theorem we may define φ_n recursively by

$$\varphi_n(x) = \frac{1}{1 - e^{i\alpha\pi(\lambda+n+\frac{1}{2})}} \int_{\gamma(x)} K(x,t)\varphi_{n-1}(t)dt$$

where $\gamma(x)$ is the circular arc $|t| = |x|$, $\arg x \leq \arg t \leq \arg x + \alpha\pi$ as in Figure 2.

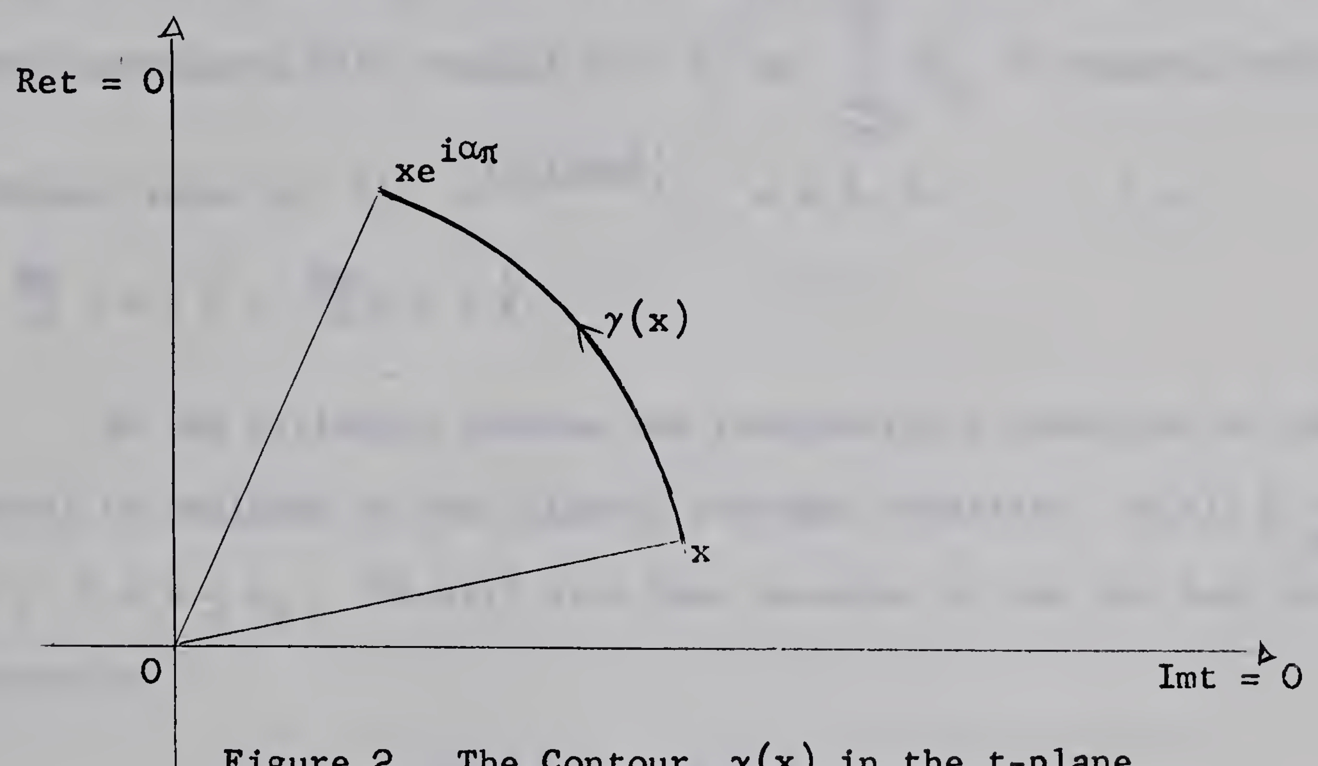

Figure 2. The Contour $\gamma(x)$ in the t-plane.

Since $\gamma(x)$ does not pass through the origin the restriction $Re\lambda \geq 0$ is no longer necessary and the φ_n are analytic functions regular everywhere with the exception of the zeros of $\prod_{s=1}^{n} (1 - e^{i\alpha\pi(\lambda+s+\frac{1}{2})})$. By induction

$$\varphi_n(x) \leq \frac{\eta(x)(G(x))^n}{\prod_{s=1}^{n} |1 - e^{i\alpha\pi(\lambda+s+\frac{1}{2})}|}$$

where

$$G(x) = |\frac{x^2}{\lambda}| \int_{\theta}^{\theta+\alpha\pi} |k^2 - V(|x|e^{i\varphi})|d\varphi \quad , \quad \theta = \arg x \; ,$$

$$= |\frac{x^2}{\lambda}| \int_{0}^{\alpha\pi} |k^2 - V(|x|e^{i\varphi})|d\varphi \; ,$$

by the periodicity of V in θ so that

$$G(x) = G(|x|) .$$

From the integrability condition on $|xV(x)|$ we must have $|x|^2|k^2 - V(x)| \to 0$ as $|x| \to 0$ so that $G(|x|) \to 0$ as $|x| \to 0$, and in particular we can choose $|x|$ sufficiently small to ensure the uniform convergence with respect to λ of $\sum_{n=0}^{\infty} \varphi_n$ in regions which do not contain zeros of $(1 - e^{i\pi(\lambda+s+\frac{1}{2})})$, $s = 1, 2, \ldots$, i.e.

$\lambda \neq \frac{2m}{\alpha} - s - \frac{1}{2}$, $\frac{2m}{\alpha} < s + \frac{1}{2}$.

In the following theorem the integrability condition on the potential is replaced by the slightly stronger condition $|V(x)| \leq \frac{c}{x^{2-\delta}}$, $\delta > 0$, $0 < x \leq x_0$. We will also have occasion to use the fact that the equation

$$\frac{d^2Y}{dx^2} - \left(\frac{\lambda^2-1/4}{x^2} + p^2x^{2\mu-2}\right)Y = 0$$

has a pair of solutions $x^{\frac{1}{2}}J_{\pm\frac{\lambda}{\mu}}\left(\frac{ipx^{\mu}}{\mu}\right)$.

<u>Theorem 2.9</u>

For any continuous potential $V(x)$ such that $|V(x)| < \frac{c}{x^{2-\delta}}$, $\delta > 0$, $0 < x \leq R$, the function $\varphi(\lambda,k;\, x)$ is regular analytic throughout the λ-plane except for possible singularities in $\mathrm{Re}\lambda \leq 0$.

The proof of this theorem is very like that of Theorem 2.6. Let

$$F(x) = k^2 - \frac{\lambda^2-1/4}{x^2} - V(x)$$

$$F_1(x) = V(x) - k^2$$

and

$$F_2(x) = V(x) - k^2 - \lambda^2 x^{2\mu-2} .$$

We may take

$$Y_1(x) = x^{-\lambda+\frac{1}{2}}$$

and

$$Y_2(x) = \Gamma(-\frac{\lambda}{\mu} + 1)(\frac{i\lambda}{2\mu})^{\frac{\lambda}{\mu}} x \, J_{-\frac{\lambda}{\mu}}(\frac{i\lambda x^{\mu}}{\mu})$$

so that

$$Y_2(x) = x^{-\lambda+\frac{1}{2}}(1 + O(x^2))$$

and

$$\frac{d}{dx} Y_2(x) = (-\lambda+\tfrac{1}{2})x^{-\lambda+\frac{1}{2}}(1 + O(x^2)) , \quad \text{as} \quad x \to 0 .$$

Taking $x_1 = x_2 = 0$ we proceed as in Theorem 2.6 and get an integral equation for y, a solution to the radial Schrödinger equation:

$$y(x) = f(x) + \int_0^x K(x,t)y(t)dt$$

where

$$f(x) = \frac{Y_2(x) - Y_1(x)}{W(Y_1(x),Y_2(x))}$$

$$= \frac{1}{2\mu} x^{\lambda+\frac{1}{2}}(1 + O(x^2)), \quad \text{as} \quad x \to 0$$

and

$$K(x,t) = \frac{Y_2(x)Y_1(t)F_1(t) - Y_1(x)Y_2(t)F_2(t)}{W(Y_1(x),Y_2(x))}$$

$$= \frac{1}{2\mu} \frac{x^{\lambda+\frac{1}{2}}}{t^{\lambda+\frac{1}{2}}} \left\{(1 - (\tfrac{t}{x})^{2\mu})(V(t)-k^2)(1 + O(x^2)) + \frac{\lambda^2\mu}{\mu-\lambda} \frac{t^{2\mu-1}}{x^{2\mu}} (1 + O(x^2)) \right\}, \quad \text{as} \quad x \to 0 .$$

As in Theorem 2.6, away from singularities in the λ-plane of f and k

$$|f(x)| \leq \eta(x)$$

and

$$|K(x,t)\eta(t)| \leq \eta(x)g(t)$$

uniformly with respect to μ and λ where

$$\eta(x) = Nx^{\mathrm{Re}\lambda+\frac{1}{2}}$$

and

$$g(t) = M(t^{\delta-1} + \frac{t^{2\mu-1}}{x^{2\mu}}) , \delta > 0 ,$$

so that if $\mu > M$ and $\frac{1}{2\mu} > \frac{x^\delta}{\delta}$ lemma 2.4 applies. The function $y(x)$ is identified by its behaviour as $x \to 0$ to be a multiple of $\varphi(\lambda,k; x)$ so that $\varphi(\lambda,k; x)$ is a regular analytic function throughout the λ-plane with the exception of possible singularities in $\mathrm{Re}\lambda \leq 0$ at singular points of $\int_0^{x_0} K(x_0,t) \frac{f(t)}{f(x_0)} dt$. Again these singularities do not have to be isolated.

§2.5 Analyticity of $f^{(1,2)}(\lambda,k)$

To apply the results of Theorems 2.4 - 2.9 to the Jost functions $f^{(1,2)}(\lambda,k)$ we recall the remark at the beginning of §2.4 that $f^{(i)}(\lambda,k)$ $(i = 1$ or $2)$ is an analytic function of λ or k in any region in which $\varphi(\lambda,k; x)$ and $F^{(i)}(\lambda,k; x)$ are both analytic. Since $(\frac{2}{k})^\lambda \varphi(\lambda,k; x)$ is an entire function of k and $F^{(i)}(\lambda,k; x)$ is an entire function of λ, $f^{(i)}(\lambda,k)$ has the same analyticity regions in λ as $\varphi(\lambda,k; x)$ and in k as $F^{(i)}(\lambda,k; x)$. We collect the results for $f^{(1,2)}(\lambda,k)$ in Theorem 2.10.

Theorem 2.10

The Jost functions $f^{(1,2)}(\lambda,k)$ are:

(i) regular analytic functions of k in $0 \leq \arg k \leq \pi$ and $-\pi \leq \arg k \leq 0$ respectively if V is as in Theorem 2.4 ,

(ii) regular analytic functions in $-\theta \leq \arg k \leq \pi + \theta_2$ and $-(\pi + \theta_1) \leq \arg k \leq \theta_2$ respectively, V as in Theorem 2.5 ,

(iii) regular analytic functions throughout the k-plane, V as in Theorem 2.6, except for the possible singularities given in that theorem.

(iv) regular analytic functions of λ , $\mathrm{Re}\lambda \geq 0$, V as in Theorem 2.7 ,

(v) regular analytic functions of λ throughout the λ-plane except for isolated singularities in $\mathrm{Re}\lambda < 0$, V as in Theorem 2.8 ,

(vi) regular analytic functions of λ throughout the λ-plane, V as in Theorem 2.9 , except for the possible singularities given in that theorem.

We close this chapter with the remark that in any result where continuity only was assumed on the potential that this condition can be relaxed to one of piecewise continuity since the integral equation (2.46) ensures the continuity of φ and its derivative.

Theorem 4.9

The Lamé functions [illegible] are

(i) [illegible] analytic functions of [illegible]

(ii) [illegible] analytic functions of [illegible]

(iii) regular analytic functions throughout the [illegible] in Theorem [illegible], except for the possible singularities [illegible] of these theorems.

(iv) regular analytic functions of [illegible]

(v) regular analytic functions of [illegible] in Theorem [illegible]

(vi) regular analytic functions of [illegible] in Theorem [illegible], except for the possible singularities given in these theorems.

[illegible]

CHAPTER III

ASYMPTOTIC PROPERTIES OF JOST FUNCTIONS

It was pointed out in Chapter I that the behaviour of $f^{(1,2)}(\lambda,k)$ as $|\lambda| \to \infty$ and as $|k| \to \infty$ is of some interest to physicists. This chapter uses the techniques of Langer as presented by Erdelyi [23] to discuss the asymptotic behaviour of $f^{(1,2)}(\lambda,k)$ when $|\lambda| + |k| \to \infty$ and $|\frac{\lambda}{k}|$ is bounded away from zero and infinity.

§3.1 Introduction

Many authors have discussed the asymptotics of the Jost functions. Of particular interest is the work of Calogero [15, 16, 17] who discusses the case $|\lambda| \to \infty$ which is also dealt with by Barut and Dilley [5] for a more restricted class of potentials. Sartori discusses the case $\lambda = \frac{1}{2}$ (S-waves), $|k| \to \infty$. Throughout this chapter the conditions on the potential, unless otherwise stated, will be piecewise continuity with (2.34) and (2.35).

Consider again the radial Schrodinger equation

$$\frac{d^2y}{dx^2} + \left(k^2 - \frac{\lambda^2-1/4}{x^2} - V(x)\right) y = 0 . \tag{3.1}$$

When $|\lambda| + |k| \to \infty$ and $|k^2 - \frac{\lambda^2}{x^2}| >> |V(x) - \frac{1}{4x^2}|$ one expects solutions to (3.1) to be closely approximated by solutions to $\frac{d^2y}{dx^2} + \{k^2 - \frac{\lambda^2}{x^2}\} y = 0$ the only difficulty arising at $x = \frac{\lambda}{k}$.

Furthermore it is not difficult to show that if $|x| \leq |\frac{\lambda}{k}| - c$ that

$y \approx Ax^{\lambda+\frac{1}{2}} + Bx^{-\lambda+\frac{1}{2}}$ and $|x| \geq |\frac{\lambda}{k}| + c, \quad c > 0, \quad y \approx Ce^{ikx} + De^{-ikx}$,

i.e. solutions to $\frac{d^2y}{dx^2} - \frac{\lambda^2-1/4}{x^2} y = 0$ and $\frac{d^2y}{dx^2} + k^2y = 0$,

respectively. It is the transition from power-like behaviour to exponential-like behaviour which gives rise to difficulties, i.e. what C and D correspond to a given A and B ?

A problem similar to the present one has received considerable attention in the literature - namely the Sturm-Liouville problem of discussing the asymptotic behaviour of solutions to the differential equation

$$\frac{d^2y}{dx^2} + (\lambda^2 p(x) + r(x,\lambda))\, y = 0 ,$$

$a < x < b$, $|\lambda| \to \infty$ and $|\lambda^2 p(x)| >> |r(x,\lambda)|$. Of particular interest here is the case when $p(x)$ changes sign at some point $x_o \epsilon (a,b)$. Our approach to the Jost function problem is essentially that contained in Erdelyi's [23] work on the Sturm-Liouville problem.

§3.2 Airy Functions

The simplest equation with a transition point is

$$\frac{d^2w}{dz^2} + zw = 0 . \tag{3.2}$$

This equation has solutions

$$w = Ai(e^{i\pi}\omega^m z) \quad \text{and} \quad w = Bi(e^{i\pi}\omega^m z)$$

where Ai and Bi are Airy functions and $\omega = e^{2\pi i/3}$, $(m = 0, \pm 1, \pm 2, \ldots)$.

In what follows some of the properties of Ai will be needed.*

Ai(z) is an entire function whose zeros are all real and negative.

$$(3.3)\qquad \mathrm{Ai}(e^{i\pi}z) = 1/3\sqrt{z}[J_{1/3}(\zeta) + J_{-1/3}(\zeta)],$$

where J is a Bessel function and $\zeta = 2/3\, z^{3/2}$ (principal branch).

$$(3.4)\qquad \mathrm{Ai}(\omega^m z) + \omega \mathrm{Ai}(\omega^{m+1}z) + \omega^2 \mathrm{Ai}(\omega^{m+2}z) = 0$$

$$(3.5)\qquad \mathrm{Ai}(z) = \tfrac{1}{2}\pi^{-\frac{1}{2}} z^{-\frac{1}{4}} e^{-\zeta}(1 + O(\zeta^{-1})), \quad \text{as } z \to \infty, \ -\pi < \arg z < \pi$$

$$(3.6)\qquad \mathrm{Ai}(z) = \tfrac{1}{2}\pi^{-\frac{1}{2}} z^{-\frac{1}{4}}\{e^{-\zeta}(1 + O(\zeta^{-1})) + ie^{\zeta}(1 + O(\zeta^{-1}))\}$$
$$\text{as } z \to \infty, \ \pi/3 < \arg z < 5\pi/3$$

$$(3.7)\qquad \mathrm{Ai}(z) = \tfrac{1}{2}\pi^{-\frac{1}{2}} z^{-\frac{1}{4}}\{e^{-\zeta}(1 + O(\zeta^{-1})) - ie^{\zeta}(1 + O(\zeta^{-1}))\}$$
$$\text{as } z \to \infty. \ -5\pi/3 < \arg z < \pi/3 .$$

From the asymptotic properties it follows that the functions

$$(3.8)\qquad (1 + |z|^{\frac{1}{4}})e^{\zeta}\mathrm{Ai}(z), \qquad -\pi \le \arg z \le \pi$$

$$(3.9)\qquad (1 + |z|^{\frac{1}{4}})e^{-\zeta}\mathrm{Ai}(z), \qquad \pi \le |\arg z| \le 5\pi/3$$

are bounded.

§3.3 Application to Jost Functions

The change of variables

$$z = \psi(x) \quad \text{and} \quad w = (\psi'(x))^{\frac{1}{2}}\, Y$$

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* Antosiewicz, H. A., "Bessel Functions of Fractional Order", U.S. Dept. of Comm., Natl. Bureau of Standards App. Math. Ser. 55 (1964).

transforms the Airy equation (3.2) into

(3.10) $$\frac{d^2Y}{dx^2} + (\psi(\psi')^2 + \tfrac{1}{2}\{\psi,x\})Y = 0$$

where

$$\{\psi,x\} = (\psi'''/\psi') - 3/2(\psi''/\psi')^2$$

is the Schwarzian derivative of ψ with respect to x. If we let $\psi(\psi')^2 = k^2 - \frac{\lambda^2}{x^2}$ then the radial Schrödinger equation (3.1) may be written

(3.11) $$\frac{d^2y}{dx^2} + (\psi(\psi')^2 + \tfrac{1}{2}\{\psi,x\})y = F(x)y$$

where $F(x) = \frac{1}{2}\{\psi,x\} + V(x) - \frac{1}{4x^2}$. From lemma 2.2 it follows that any solution to (3.11) satisfies the integral equation

(3.12) $$y(x) = Y(x) + \int_c^x K(x,t)y(t)dt ,$$

and vice versa, where

$$K(x,t) = \frac{F(t)}{\Delta}(Y_j(x)Y_i(t) - Y_i(x)Y_j(t)) ,$$

Y, Y_i and Y_j being solutions to (3.10) such that $W(Y_i, Y_j) = \Delta \neq 0$, and $0 \leq x \leq \infty$.

Since $\psi(\psi')^2 = k^2 - \frac{\lambda^2}{x^2}$ we have

$$2/3(\psi(x))^{3/2} = \alpha(x) = \int_{\frac{\lambda}{k}}^x (k^2 - \frac{\lambda^2}{t^2})^{\frac{1}{2}} dt$$

$$= kx(1 - \frac{\lambda^2}{k^2x^2})^{\frac{1}{2}} + i\lambda \log(\frac{\lambda}{kx} + i(1 - \frac{\lambda^2}{k^2x^2})^{\frac{1}{2}}) ,$$

here we choose that branch of the square-root which is positive when $\frac{\lambda^2}{k^2x^2} < 1$ and the principal branch of the logarithm. From the chain rule for Schwarzian derivatives,

$$\{\psi,x\} = \{\psi,\alpha\}(\alpha')^2 + \{\alpha,x\} ,$$

we get

$$F(x) = \frac{5}{36}\left(\frac{\alpha'}{\alpha}\right)^2 + V(x) - \frac{4k^2\lambda^2 + k^4x^2}{4(k^2x^2 - \lambda^2)^2} .$$

When $|\frac{\lambda}{k}| > |x| \to 0$

$$\alpha(x) = -i\lambda[\log(\frac{kx}{2\lambda}) + 1 + O(\frac{k^2x^2}{\lambda^2})]$$

$$\alpha'(x) = \frac{-i\lambda}{x}[1 + O(\frac{k^2x^2}{\lambda^2})] ,$$

so that $F(x) = O((x \log x)^{-2}) + O(V(x))$, as $x \to 0$. Similarly when $|\frac{\lambda}{k}| < |x| \to \infty$

$$\alpha(x) = kx[1 - \frac{\pi\lambda}{2kx} + O(\frac{\lambda^2}{k^2x^2})]$$

$$\alpha'(x) = k[1 + O(\frac{\lambda^2}{k^2x^2})]$$

and $F(x) = O(x^{-2}) + O(V(x))$, as $x \to \infty$. Since $\psi(\psi')^2 = k^2 - \frac{\lambda^2}{x^2}$ we see that $\psi'(x) \neq 0$, $x \neq \frac{\lambda}{k}$ and $F(x)$ is defined everywhere except possibly $x = \frac{\lambda}{k}$. As $x \to \frac{\lambda}{k}$, $(k^2x^2 - \lambda^2)^2\frac{5}{36}(\frac{\alpha'}{\alpha})^2 \to \frac{5}{4}k^2\lambda^2 + \frac{1}{2}k^3\lambda(x - \frac{\lambda}{k})$ so that

$$F(\frac{\lambda}{k}) = V(\frac{\lambda}{k}) + O(\frac{k^2}{\lambda^2})$$

and hence

$$[(x\log(2 + \frac{1}{x}))^{-2} + |V(x)|]^{-1} F(x) \tag{3.13}$$

is bounded uniformly with respect to λ , k and x .

We will use three solutions to the equation (3.10):

(3.14) $$Y_m = (\psi')^{-\frac{1}{2}} \mathrm{Ai}(e^{i\pi}\omega^m\psi) , \quad m = 0, 1, 2,$$

and these may be written

$$Y_m = (\psi')^{-\frac{1}{2}} \mathrm{Ai}(\theta_m), \quad m = 0, 1, 2,$$

where

$$\theta_o = e^{i\pi}\psi , \quad \theta_1 = e^{-i\pi}\omega\psi , \quad \theta_2 = e^{i\pi}\omega^{-1}\psi .$$

From (3.8) and (3.9)

(3.15) $$|Y_m| \leq \eta_m$$

where

$$\eta_m = \Psi \exp(-\mathrm{Re}\ 2/3\ \theta_m^{3/2}) , \quad -\pi \leq \arg \theta_m \leq \pi ,$$

$$\eta_m = \Psi \exp(\mathrm{Re}\ 2/3\ \theta_m^{3/2}), \quad \pi \leq |\arg \theta_m| \leq 5\pi/3$$

and $\Psi = C|\psi'|^{-\frac{1}{2}}(1 + |\psi|^{\frac{1}{4}})^{-1}$. Finally we remark that

$$2/3\ \theta_o^{3/2} = -i\alpha , \quad 2/3\ \theta_1^{3/2} = -i\alpha \quad \text{and} \quad 2/3\ \theta_2^{3/2} = i\alpha .$$

Before proceeding further we give the following lemma.

Lemma 3.1

(i) If $|\frac{\lambda}{k}| \leq x < \infty$ then $\mathrm{Im}\alpha(x)$ is non-increasing (for increasing x) if $\arg \lambda = \arg k$ and $\mathrm{Im}k \leq 0$ and non-decreasing if $\mathrm{Im}k \geq 0$. For unrestricted λ , $\mathrm{Im}\alpha(x)$ is non-increasing if $-3\pi/4 \leq \arg k \leq -\pi/4$ and non-decreasing if $\pi/4 \leq \arg k \leq 3\pi/4$.

(ii) If $0 < x \leq |\frac{\lambda}{k}|$ then $\mathrm{Im}\alpha(x)$ is non-increasing (for increasing x) if $\arg\lambda = \arg k$, $\mathrm{Re}\lambda \geq 0$ and for unrestricted k , $\mathrm{Im}\alpha(x)$ is non-increasing if $|\arg\lambda| < \pi/4$.

We indicate the proof of (i) and that of (ii) is similar.

Since

$$\alpha(x) = \int_{\frac{\lambda}{k}}^{x} (k^2 - \frac{\lambda^2}{t^2})^{\frac{1}{2}} dt$$

$$\frac{d}{dx}\mathrm{Im}\alpha(x) = \mathrm{Im}(k^2 - \frac{\lambda^2}{x^2})^{\frac{1}{2}}$$

$$= \mathrm{Im}[k(1 - \frac{\lambda^2}{k^2x^2})^{\frac{1}{2}}] ,$$

If $\arg\lambda = \arg k$ then $\frac{d}{dx}\mathrm{Im}\alpha(x)$ has the same sign as $\mathrm{Im}k$, since $\frac{\lambda^2}{k^2x^2} \leq 1$, and this gives the first part of (i) . The second part of (i) follows from the fact that $-\pi/4 \leq \arg(1 - \frac{\lambda^2}{k^2x^2})^{\frac{1}{2}} \leq \pi/4$, since $|\frac{\lambda^2}{k^2x^2}| \leq 1$.

We are now in a position to define some solutions to the radial Schrödinger equation (3.11) and to examine their asymptotic behaviour as $|\lambda| + |k| \to \infty$. Three solutions are given by

$$(3.16) \qquad y_m(x) = Y_m(x) + \int_{x_m}^{x} K(x,t)y_m(t)dt , \quad m = 0, 1, 2,$$

where

$$x_0 = 0 , x_1 = x_2 = \infty ,$$

and from (3.4) and (3.12)

$$K(x,t) = 2\pi\omega^{\frac{1}{4}} F(t)(Y_o(x)Y_1(t) - Y_1(x)Y_o(t))$$

$$= 2\pi\omega^{-\frac{3}{4}} F(t)(Y_1(x)Y_2(t) - Y_2(x)Y_1(t))$$

$$= 2\pi\omega^{-\frac{1}{4}} F(t)(Y_2(x)Y_o(t) - Y_o(x)Y_2(t)) .$$

<u>Theorem 3.1</u>

As $|\lambda| + |k| \rightarrow \infty$,

(3.17) $$y_m = Y_m + \eta_m O((|\lambda| + |k|)^{-1}) , \quad m = 0, 1, 2,$$

uniformly with respect to x when $0 < |\frac{\lambda}{k}| < \infty$,

$m = o$, $|\arg \lambda| \leq \pi/4$, $\pi/4 \leq |\arg k| \leq 3\pi/4$

$m = 1$, $|\arg \lambda| \leq \pi/4$, $\pi/4 \leq \arg k \leq 3\pi/4$

and

$m = 2$, $|\arg \lambda| \leq \pi/4$, $-3\pi/4 \leq \arg k \leq -\pi/4$.

Also (3.17) holds when $\arg \lambda = \arg k$ and

$m = 0$, $|\arg \lambda| \leq \pi/2$

$m = 1$, $0 \leq \arg k \leq \pi$

and

$m = 2$, $-\pi \leq \arg k \leq 0$.

This result is essentially a generalisation of Erdélyi's [23] results for Bessel functions $J_\lambda(\lambda x)$ and Hankel functions $H_\lambda^{(1,2)}(\lambda,x)$ where λ and x are real and $\lambda \rightarrow \infty$.

Proof for $m = 0$:

In the integral equation (3.16)

$$|K(x,t)| \leq M\Psi(x)\Psi(t)\ [\exp(\alpha(x)-\alpha(t)) + \exp(\alpha(t)-\alpha(x))]F(t)$$

where M is independt of λ, k, x and t. When $0 \leq t \leq x \leq |\frac{\lambda}{k}|$, from (3.15),

$$Y_o(x) \leq \eta_o(x) = \Psi(x)\exp(-\mathrm{Im}\,\alpha(x)),\ \mathrm{Re}\lambda \geq 0 .$$

Since, from Lemma 3.1, $\mathrm{Im}\,\alpha(x)$ is non-increasing if $|\arg\lambda| \leq \frac{\pi}{4}$ we find

$$(3.18) \qquad |K(x,t)\eta_o(t)| \leq \eta_o(x)2M(\Psi(t))^2F(t)$$

$$= \eta_o(x)g(t)$$

where $g = 2M(\Psi)^2F$.

Similarly when $x > |\frac{\lambda}{k}|$, $0 \leq t \leq x$,

$$(3.19) \qquad |K(x,t)\eta_o(t)| \leq \eta_o(x)g(t)$$

where in addition to $|\arg\lambda| < \frac{\pi}{4}$ we must now make the restriction $\frac{\pi}{4} < |\arg k| < \frac{3\pi}{4}$ to use lemma 3.1.

Since

$$(\Psi)^2 = c|\psi'|^{-1}(1 + |\psi|^{\frac{1}{4}})^{-2}$$

$$\leq c|\psi'(\psi)^{\frac{1}{2}}|^{-1}$$

$$= c\left|k^2 - \frac{\lambda^2}{x^2}\right|^{-\frac{1}{2}}$$

and from (3.13) and the conditions (3.34) and (2.35) on the potential

$$G(x) = \int_0^x g(t)dt = O((|\lambda| + |k|)^{-1}) \tag{3.20}$$

uniformly with respect to x in the (λ,k) regions described in the statement of the theorem. Hence, from (3.18) - (3.20) and lemma 2.3

$$y_0 = Y_0 + \eta_0 O(|\lambda| + |k|)^{-1}) , \quad 0 < |\frac{\lambda}{k}| < \infty ,$$

uniformly with respect to x. The proof for $m = 1$ and $m = 2$ is similar.

It now remains to identify the solutions y_m to the radial Schrödinger equation. From the behaviour of Y_0 as $x \to 0$,

$$y_0(x) = \tfrac{1}{2}\pi^{\frac{1}{2}}\lambda^{-\frac{1}{2}}e^{\lambda}(\frac{kx}{2\lambda})^{\lambda} x^{\frac{1}{2}}[1 + O((\log x)^{-1} + O(\int_0^x tV(t)dt)] , \tag{3.21}$$

and from Theorem 2.2 ,

$$\varphi(\lambda,k, x) = (\frac{k}{2})^{\lambda}(\Gamma(\lambda+1))^{-1}x^{\lambda+\frac{1}{2}}[1 + O(x^2) + O(\int_0^x tV(t)dt)] . \tag{3.22}$$

Comparison of (3.21) and (3.22) gives

$$\varphi = 2\pi^{\frac{1}{2}}\lambda^{\lambda+\frac{1}{2}}e^{-\lambda}(\Gamma(\lambda+1))^{-1}y_0 , \tag{3.23}$$

and similarly

$$F^{(1)} = 2^{3/2}e^{-i\pi/3}y_1 , \tag{3.24}$$

and

$$F^{(2)} = 2^{3/2}e^{i\pi/3}y_2 . \tag{3.25}$$

It should be remarked that since $\lambda \to \infty$ (3.23) simplifies, by Stirling's formula, to

$$\varphi = (2^{\frac{1}{2}} + O(\frac{1}{\lambda}))y_0 .$$

Theorem 3.1 and the equations (3.23) - (3.25) give the following results.

Theorem 3.2

As $|\lambda| + |k| \to \infty$,

$$(3.26)\qquad \varphi = 2\pi^{\frac{1}{2}}\lambda^{\lambda+\frac{1}{2}}e^{-\lambda}(\Gamma(\lambda+1))^{-1}Y_o + \eta_o\, O((|\lambda| + |k|)^{-1}) ,$$

$$(3.27)\qquad F^{(1)} = 2^{3/2}e^{-i\pi/3}\, Y_1 + \eta_1 O((|\lambda| + |k|)^{-1})$$

$$(3.28)\qquad F^{(2)} = 2^{3/2}e^{i\pi/3}\, Y_2 + \eta_2 O((|\lambda| + |k|)^{-1})$$

and the bounds implied by the Hardy O's are uniform with respect to x , the parameters λ and k being restricted as in Theorem 3.1 for $m = 0$, $m = 1$ and $m = 2$, respectively.

We use Theorems 2.3 and 3.2 to get asymptotic expressions for the Jost functions. As $|\lambda| + |k| \to \infty$, $0 < |\frac{\lambda}{k}| < \infty$,

$$(3.29)\qquad f^{(1)}(\lambda,k) = 1 + (2\pi)^{3/2}e^{i\pi/6}\lambda^{\lambda+\frac{1}{2}}e^{-\lambda}(\Gamma(\lambda+1))^{-1}\int_o^\infty V(x)Y_o(x)Y_1(x)dx + \Sigma^{(1)}(\lambda,k) .$$

and

$$(3.30)\qquad f^{(2)}(\lambda,k) = 1 + (2\pi)^{3/2}e^{-i\pi/6}\lambda^{\lambda+\frac{1}{2}}e^{-\lambda}(\Gamma(\lambda+1))^{-1}\int_o^\infty V(x)Y_o(x)Y_2(x)dx + \Sigma^{(2)}(\lambda,k) .$$

In (3.29)

$$|\Sigma^{(1)}(\lambda,k)| \leq C\int_o^\infty V(x)\eta_o(x)\eta_1(x)dx\, O((|\lambda| + |k|)^{-1})$$

$$= O((|\lambda| + |k|)^{-2})$$

provided $|\arg \lambda| \leq \pi/4$, $\pi/4 \leq \arg k \leq 3\pi/4$, or if $\arg \lambda = \arg k$, $0 \leq \arg k \leq \pi/2$. Similarly in (3.30)

$$\Sigma^{(2)}(\lambda,k) = O((|\lambda| + |k|)^{-2})$$

provided $|\arg \lambda| \leq \pi/4$, $-3\pi/4 \leq \arg k \leq -\pi/4$, or if $\arg \lambda = \arg k$, $-\pi/2 \leq \arg k \leq 0$.

Simpler expressions result if we use Theorem 3.2 for the free Schrödinger equation and equations (3.29) and (3.30) .

Theorem 3.3

As $|\lambda| + |k| \to \infty$,

$$f^{(1,2)}(\lambda,k) = 1 \pm i\pi/2 \int_0^\infty V(x)F_0^{(1,2)}(x)\varphi_0(x)dx + \epsilon^{(1,2)}(\lambda,k) \tag{3.31}$$

or

$$f^{(1,2)}(\varphi,k) = 1 \pm i\pi/2 \int_0^\infty xV(x)H_\lambda^{(1,2)}(kx)J_\lambda(kx)dx + \epsilon^{(1,2)}(\lambda,k) \tag{3.32}$$

where $\epsilon^{(1,2)}$ are $O((|\lambda| + |k|)^{-2})$ in the same regions as those given for $\Sigma^{(1,2)}$, respectively.

Remarks.

(i) It had been hoped that Erdélyi-Wyman-type [25] results, i.e. without the restriction that $|\frac{\lambda}{k}|$ be bounded away from 0 and ∞ , could be obtained by this procedure; however it seems evident that a more sophisticated technique is needed to remove this restriction.

(ii) The admissible regions for the approach of λ and k to infinity can be extended for analytic potentials like those introduced in Chapter II , e.g. by integrating along a ray system in (3.32). It seems reasonable, however, that (3.32) should hold in $\mathrm{Re}\lambda \geq 0$ and $\mathrm{Im}k \geq 0$ and ≤ 0 , respectively, for quite general potentials, but the estimate on the error needs modification.

(iii) Equations (3.29) and (3.30) imply that under the given conditions the Jost functions approach their unperturbed values, i.e. $f^{(1,2)}(\lambda,k) \rightarrow 1$.

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